26 Convergence

(*These notes are somewhat different from the version handed out in class – even the title changed!*

26.1 Recap: convergence

Recall that we saw that the sequence

$$\{G^0, G^1, \ldots, G^k, \ldots\},$$

will converge to zero if $\rho(G) < 1$.

Recall that we can write iterative schemes, such as Jacobi or Gauss-Seidel as

$$\mathbf{x}^{(k+1)} = G\mathbf{x}^{(k)} + f,$$

If $A\mathbf{x} = \mathbf{b}$, then the sequence generated satisfies

$$\mathbf{x} - \mathbf{x}^{(k)} = G^k(x - x^{(0)}).$$

So, if $\rho(G) < 1$, the iteration will converge (to the solution of $A\mathbf{x} = \mathbf{b}$) for any initial guess $\mathbf{x}^{(0)}$.

Example 26.1. If the matrix A in the system $A\mathbf{x} = \mathbf{b}$ is strictly diagonally dominant, then the associated Jacobi iteration will converge.

In general, it is not so easy to work out $\rho(G)$ for a given matrix G. However, since $\rho(G) \leqslant \|G\|$ for any consistent matrix norm (see Exercise 26.2), then we can take $\|G\|$ is a sufficient (but not necessary) condition for convergence.

26.2 The Jordan canonical form

(I'd thought I'd gloss over this topic, but have realised that it is too useful). In our "proof" of 25.2, we only considered a special case. For the general case, we need the notion of the *Jordan canonical form.*

Recall that matrices A and B are *similar* if there exists an invertible matrix X such that $A = XBX^{-1}$. It is clear that A and B have the same eigenvalues.

Definition 26.2.
1. An eigenvalue λ of A has *algebraic multiplicity* μ if it is a root of multiplicity μ of the characteristic polynomial of A.
2. If λ has multiplicity 1 it is *simple.*
3. The *geometric multiplicity*, γ, of an eigenvalue λ is the number of linearly independent eigenvalues of A.
4. An eigenvalue is *semisimple* if its geometric multiplicity is equal to its algebraic multiplicity. If it is not semisimple, it is *degenerate.*

Definition 26.3 (Jordan Canonical Form)**.** For any matrix A there are matrices X and J such that $A = X^{-1}JX$, and J has the form

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{pmatrix}.$$

where the J_i are block each associated with a distinct eigenvalue λ_i of A. Each of these blocks can be expressed as γ_i "Jordan" blocks where γ_i is the geometric multiplicity of λ_i. These blocks are

$$J_i = \begin{pmatrix} J_{i1} & & & \\ & J_{i2} & & \\ & & \ddots & \\ & & & J_{i\gamma_i} \end{pmatrix}.$$

where J_{ik} is the bidiagonal matrix

$$J_{ik} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \\ & & & \lambda_i \end{pmatrix}$$

26.3 Exercises

Exercise 26.1. Show that, if A is diagonally dominant, then the Gauss-Seidel iteration converges.

Exercise 26.2. Show that, if $\|\cdot\|$ is a consistent matrix norm, then $\rho(G) \leqslant \|G\|$.