27 The JCF, and applications

27.1 Recall: defective matrices

In Lecture 26 we saw that the *algebraic multiplicity* of an eigenvalues λ, is the multiplicity of the associated root in the characteristic polynomial. We denote this $\mu(\lambda)$. This eigenvalue's *geometric multiplicity*, $\gamma(\lambda)$ is the number of linearly independent eigenvectors associated with it. Always, $\gamma(\lambda) \leqslant \mu(\lambda)$ (see Exercise 27.1). If $\gamma(\lambda) < \mu(\lambda)$, the eigenvalue is *degenerate*. Moreover, the matrix is *defective*: that is, it does not have a full set of linearly independent eigenvectors.

If a matrix, $A \in \mathbb{R}^{m\times m}$ is *not* defective, then we say it is *diagonalisable*. This means that, since it has m linearly independent eigenvectors, $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$, we can form the matrix $X = (\mathbf{x}_1|\cdots|\mathbf{x}_m)$, and thus write

$$A = X^{-1}\Lambda X,$$

where Λ is a diagonal matrices whose entries are the eigenvalues of A

As we saw in class, this diagonalisation – when it exists – is a useful theoretical tool. But often, of course, it does not exist.

27.2 A degenerate matrix

The simplest, and most important, example of a degenerate matrix is

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

It is easy to see that its only eigenvalue is α, but that $\mu(\alpha) = 2$ and $\gamma(\alpha) = 1$. Similarly, the matrix

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$$

is degenerate. However, it is an easy matrix to work with: for example we can tell just by looking at it what the eigenvalues are, and what their multiplicities are.

27.3 Recall: The Jordan canonical form

The JCF is a similarity transform that reduces a matrix to a block-diagonal one. I don't reproduce it here: you can see it in the notes from Lecture 26. But we can not get a sense of what it means.

- Each of the bidiagonal matrices J_{ik} is associated with a single eigenvector of the matrix.
- Each of the matrices J_i is associated with a single eigenvalue, λ_i; and J_i has $\mu(\lambda_i)$ rows and columns.

Every matrix (even degenerate ones), can be written in the form $A = X^{-1}JX$. This has many applications, e.g., we can see immediately that if a matrix has distinct eigenvalues, then it is diagonalisable.

27.4 General convergence

Let $\mathbf{d}^{(k)} = \mathbf{x} - \mathbf{x}^{(k)}$.

Definition 27.1. The *General convergence factor* is

$$\phi = \lim_{k\to\infty} \left(\max_{x_0 \in \mathbb{R}^m} \frac{\|\mathbf{d}^{(k)}\|}{\|\mathbf{d}^{(0)}\|} \right)^{1/k}.$$

We can show that $\phi = \rho(G)$. This makes use of Exercise 27.2.

27.5 Exercises

Exercise 27.1. It is a fact (which can be established with some tedium from the usual (Laplace) expansion for the determinant) that if the matrix M is partitioned as

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

then $\det(M) = \det(A)\det(D)$. Use this to show that the geometric multiplicity of an eigenvalue is less than or equal to its algebraic multiplicity.

Exercise 27.2. (Tricky.) Use the Jordan canonical form to show that, for any subordinate matrix norm,

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A).$$