28 Regular splittings

28.1 Preliminary results

The next topic we want to look at is the notion of a *regular splitting.* But before then, we need some basic ideas from linear algebra.

Recall that $\rho(A)$ is the spectral radius of the matrix A.

Theorem 28.1. *If* $\rho(A) < 1$*, then the matrix* $B = I - A$ *is nonsingular.*

Theorem 28.2 (Theorem 1.11 of [Saa03])**.** *The series*

$$\sum_{k=0}^{\infty} A^k$$

converges if and only if $\rho(A) < 1$*, in which case* $I - A$ *is nonsingular, and*

$$(I - A)^{-1} = \lim_{p\to\infty} \sum_{k=0}^{p} A^k$$

28.2 Nonnegative matrices

(See [Saa03, Section 1.10]).

Definition 28.3. If the matrices A and B are of the same size, when we write $A < B$, we mean $a_{ij} < b_{ij}$ for each i, j. Similarly,

$$A \leqslant B \qquad \Longleftrightarrow \qquad a_{ij} \leqslant b_{ij}\ \forall i, j.$$

Use O to designate the zero matrix. When we say "*the matrix* A *is nonnegative*", we mean $O \leqslant A$. When we say "*the matrix* A *is positive*", we mean $O < A$.

Definition 28.4. A matrix A is *reducible* is there exists a permutation matrix P such that PAP^T is block upper triangular.

A more intuitive, equivalent definition, is that the matrix A is irreducible if its graph is strongly connected.

The next result is classic, though we won't prove it.

Theorem 28.5 (Perron-Frobenius)**.** *If the matrix* A *is nonnegative, then* $\lambda = \rho(A)$ *is a eigenvalue of* A *with* $\mu(\lambda) = 1$*. There exists an associated eigenvector* u *such that* $u \geqslant 0$*.*

If, furthermore, A *is irreducible, then* $u > 0$*.*

Other important properties of nonnegative matrices include:

Theorem 28.6. *Let* A*,* B *and* C *be nonnegative matrices, with* A leq$B$*. Then*

$$AC \leqslant BC \quad \textit{and} \quad CA \leqslant CB.$$

Furthermore, $A^k \leqslant B^k$ *for* $k = 0, 1, \dots$*.*

The proof is left as Exercise 28.1.

Theorem 28.7. *Suppose that* $B \geqslant 0$*. Then* $\rho(B) < 1$ *if and only if* $I - B$ *is nonsingular and* $(I - B)^{-1} \geqslant 0$*.*

28.3 Regular splitting

Definition 28.8. The pair of matrices M and N is a *regular splitting* of the matrix A if

(a) $A = M - N$.

(b) M^{-1} exists, and

(c) both M^{-1} and N are nonnegative.

Now define the iteration

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}\mathbf{b}. \tag{28.1}$$

We want to know when this will converge. We can use Theorem 28.2 to prove the following

Theorem 28.9 (Thm 4.4 of [Saa03])**.** *Let* M*,* N *be a regular splitting of* A*. Then* $\rho(M^{-1}N) < 1$ *if and only if* A *is nonsingular and* A^{-1} *is nonnegative.*

So we now can establish when an iteration of the form (28.1) converges. These apply directly to the Jacobi and Gauss-Seidel methods. See Exercise 28.2

28.4 Exercises

Exercise 28.1. Prove Theorem 28.6.

Exercise 28.2. Recall from (25.1) that the Jacobi method can be written as

$$\mathbf{x}^{(k+1)} = D^{-1}(E + F)\mathbf{x}^{(k)} + D^{-1}b,$$

and the Gauss-Seidel as

$$\mathbf{x}^{(k+1)} = (D - E)^{-1}F\mathbf{x}^{(k)} + (D - E)^{-1}b.$$

When conditions do you need on A for these to be regular splittings?