33 Analysis of CG

(DRAFT)

We'll devote this class to proving the following result. Here, by "iterates" we mean the $\mathbf{x}^{(k)}$; the errors are $\mathbf{e}^{(k)} = A^{-1}b - \mathbf{x}^{(k)}$; and the residuals are $\mathbf{r}^{(k)} = b - A\mathbf{x}^{(k)}$. Note that $\mathbf{e}^{(k)} = A^{-1}\mathbf{r}^{(k)}$.

Theorem 33.1 (Thm 2.3.2 of [Gre97]). *Suppose that A is s.p.d. and the CG algorithm generates the sequence* $\{\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots\}$. *Then*

(a) The iterates, errors and residuals are all well-defined.

(b) The vectors $\mathbf{r}^{(k)}$, $\mathbf{e}^{(k)}$ *and* $\mathbf{p}^{(k)}$ *satisfy*

$$(\mathbf{r}^{(k+1)}, \mathbf{r}^{(j)}) = 0, \qquad (33.1)$$

$$(\mathbf{e}^{(k+1)}, A\mathbf{p}^{(j)}) = 0, \qquad (33.2)$$

$$(\mathbf{p}^{(k+1)}, A\mathbf{p}^{(j)}) = 0, \qquad (33.3)$$

for all $j \leqslant k$.

(c) CG generates the exact solution after m *iterations.*

(d) Of all the vectors in the affine space

$$\mathbf{e}^{(0)} + \mathrm{span}\{A\mathbf{e}^{(0)}, A^2\mathbf{e}^{(0)}, ; A^{k+1}\mathbf{e}^{(0)}\},$$

$\mathbf{e}^{(k+1)}$ *has the smallest* A*-norm.*