MA519 Numerical Linear Algebra: Problem Set 1

Homework exercises are marked with a ♣ .

Q1. Show that the function that maps $A \in \mathbb{R}^{n\times n}$ to $A \to |\det(A)|$ is *not* a matrix norm.

Q2. Do Exercises (i)–(iii) in Section 1.7 of Ipsen's *Numerical Matrix Analysis: Linear Systems and Least Squares*.

Q3. ♣ Do Exercises (iv)–(vii) in Section 1.7 of Ipsen's *Numerical Matrix Analysis: Linear Systems and Least Squares*.

Q4. The matrix $L \in \mathbb{R}^{n\times n}$ is called *Lower Triangular* (LT) if $l_{ij} = 0$ if $i < j$. Furthermore, if $l_{ii} = 1$, it is called *Unit Lower Triangular*. Show that the product of two LT matrices is an LT matrix.
Show that, if $L = I + N$, then N is nilpotent.
Show that

$$L^{-1} = I - N + N^2 - N^3 + \dots.$$

Deduce that L^{-1} is itself a ULT matrix.

Q5. Suppose you know the rank of the square $m \times m$ matrices A and B. What, if anything, can you say about the rank of $C = AB$? Suppose A or B have full rank, what can you say about the rank of C?

Q6. Show that if a matrix is triangular and unitary, it must be diagonal.

Q7. ♣ Suppose that $\{q_1, q_2, \dots, q_n\}$ is an orthonormal set. Show that for any vector v,

$$r = v - (q_1^\star v)q_1 - (q_2^\star v)q_2 - \cdots - (q_n^\star v)q_n,$$

is orthogonal to $\{q_1, q_2, \dots, q_n\}$.

Q8. ♣ Show that if $A \in \mathbb{C}^{m\times m}$ is an hermitian matrix, then all its eigenvalues are real. Show that if x_1 and x_2 are eigenvectors associated respectively with eigenvalues $\lambda_1 \neq \lambda_2$, then x_1 and x_2 are orthogonal.

Q9. A matrix $S \in \mathbb{C}^{m\times m}$ is *skew-hermitian* if $S^\star = -S$. Show that the eigenvalues of S are pure imaginary.

Q10. If Q is a unitary matrix, what can one say about its eigenvalues? What (if anything) can one say about its singular values?

Q11. Show that for any of subordinate matrix norm, $\|I\| = 1$. (That is, the norm of the identity matrix is 1). Is this also true of $\|\cdot\|_F$?

Q12. In Definition 4.1, we state a norm had the properties that

- $\|x\| = 0 \iff x = \mathbf{0}$.
- $\|\lambda x\| = |\lambda|\|x\|$.
- $\|x + y\| \leqslant \|x\| + \|y\|$.

It is possible to deduce from these properties that $\|x\| \geqslant 0$?

Q13. Show carefully that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms on $\mathbb{C}^m$. Suppose that $A \in \mathbb{C}^{m\times m}$ is a given invertible matrix. Is it true that $\|Ax\|_1$ is a norm on $\mathbb{C}^m$?

Q14. ♣ Show that the following inequalities hold for all vectors $x \in \mathbb{C}^m$. If possible, give a nontrivial example for which equality holds.

(a) $\|x\|_\infty \leqslant \|x\|_2$
(b) $\|x\|_2 \leqslant \sqrt{m}\|x\|_\infty$
(c) $\|x\|_2 \leqslant \|\sqrt{x\|_1\|x\|_\infty}?$
(d) $\|x\|_\infty \leqslant \|x\|_2 \leqslant \|x\|_1$

Which, if any, of these inequalities extend to the matrix norms induced by these vector norms?

Q15. Show that, for any norm on $\mathbb{R}^m$, the corresponding subordinate matrix norm of the identity matrix is 1.

Q16. Show that for any matrix $A \in \mathbb{R}^{m\times m}$,

$$\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{m} |a_{i,j}|.$$

Q17. Show that for any subordinate matrix norm, $\|A\| \geqslant |\lambda|$ for any eigenvalue λ of A.

Q18. ♣ Recall from §5.4 that a matrix norm, $\|\cdot\|$ is *consistent* (also know as *sub-multiplicative*) if

$$\|AB\| \leqslant \|A\|\|B\|.$$

(a) Show that any subordinate matrix norm on $\mathbb{C}^{m\times n}$ is consistent.
(b) Consider the function $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by
$$\|A\|_{\tilde\infty} = \max_{i,j} |a_{ij}|.$$
Show that this is a norm. Show that it is *not* consistent.
(c) Prove that the *Frobenius norm* is consistent.

Q19. From the examples of unitary matrices that we looked at in class it seems that, if Q is unitary, then

(i) $\det(Q) = 1$,
(ii) If λ is an eigenvalue of Q, then $|\lambda| = 1$.

Prove these statements, or give counter-examples to them.

Q20. ♣ Prove theorem 9.9.