7 Lecture 7: Norms

We are in processing of studying the error analysis on the finite difference method applied to problem (4.1). However, before we can complete this, we need to revise some important properties of *norms*.

First (today) we'll look at **vector** and **matrix** norms. Then we'll look at **grid** norms that were mentioned briefly during Lecture 6.

7.1 Three vector norms

When we want to consider the size of a real number, without regard to sign, we use the *absolute value*. Important properties of this function are:

(i) $|x| \geqslant 0$ for all x. (ii) $|x| = 0$ if and only if $x = 0$. (iii) $|\lambda x| = |\lambda||x|$. (iv) $|x + y| \leqslant |x| + |y|$.

This notion can be extended to vectors and matrices.

Definition 7.1. Let $\mathbb{R}^n$ be the set of all the vectors of length n of real numbers. The function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is called a **norm** on $\mathbb{R}^n$ if, for all $u, v \in \mathbb{R}^n$

1. $\|v\| \geqslant 0$,
2. $\|v\| = 0$ if and only if $v = 0$.
3. $\|\lambda v\| = |\lambda|\|v\|$ for any $\lambda \in \mathbb{R}$,
4. $\|u + v\| \leqslant \|u\| + \|v\|$ (triangle inequality).

The norms of a vector give us some information about the *size* of the vector. But there are different ways of measuring the size: you could take the absolute value of the largest entry, you cold look at the "distance" for the origin, etc... There are three important examples.

Definition 7.2. Let $\boldsymbol{v} \in \mathbb{R}^n$: $\boldsymbol{v} = (v_1, v_2, \dots, v_{n-1}, v_n)^T$.

(i) The 1-*norm* (also known as the *Taxi cab* or *Manhattan* norm) is: $\|v\|_1 = \sum_{i=1}^{n} |v_i|$.

(ii) The 2-*norm* (a.k.a. the *Euclidean norm*) is: $\|v\|_2 = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$.

Note, if $\boldsymbol{v}$ is a vector in $\mathbb{R}^n$, then

$$\boldsymbol{v}^T\boldsymbol{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\boldsymbol{v}\|_2^2.$$

(iii) The ∞-*norm* (also known as the $\max$-*norm*) is $\|v\|_\infty = \max_{i=1}^{n} |v_i|$.

Example 7.3. If $\boldsymbol{v} = (-2, 4, -4)^T$ then ...

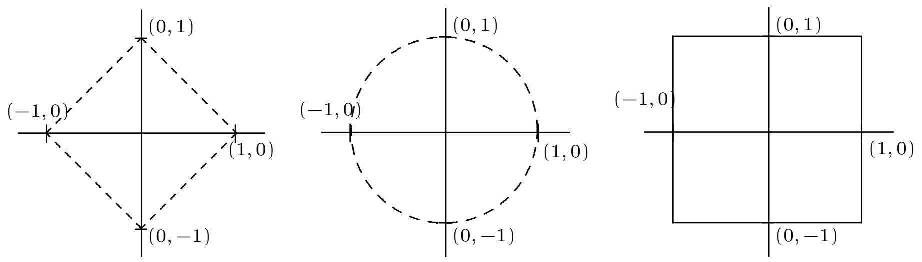

Fig. 7.1: The unit vectors in $\mathbb{R}^2$: $\|\boldsymbol{x}\|_1 = 1$, $\|\boldsymbol{x}\|_2 = 1$, $\|\boldsymbol{x}\|_\infty = 1$,

In Figure 7.1, the first diagram shows the unit ball in $\mathbb{R}^2$ given by the 1-norm: the vectors $x = (x_1, x_2)$ in $\mathbb{R}^2$ are such that $\|x\|_1 = |x_1| + |x_2| = 1$ are all found on the diamond (top left). In the second diagram, the vectors have $\sqrt{x_1^2 + x_2^2} = 1$ and so are arranged in a circle (top right). The bottom diagram gives the unit ball in $\|\cdot\|_\infty$, for which the largest component of each vector is 1.

It is easy to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms (see exercise). And it is not hard to show that $\|\cdot\|_2$ satisfies conditions (1), (2) and (3) of Definition 7.1. But it takes a little bit of effort to show that $\|\cdot\|_2$ satisfies the triangle inequality. First we need

Lemma 7.4 (Cauchy-Schwarz)**.**

$$|\sum_{i=1}^{n} u_i v_i| \leqslant \|\mathbf{u}\|_2 \|\mathbf{v}\|_2, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

The proof can be found in any text-book on analysis. Now can now apply it to show that

Lemma 7.5.

$$\|\mathbf{u}+\mathbf{v}\|_2 \leqslant \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2.$$

It follows directly that

Corollary 7.6. $\|\cdot\|_2$ *is a norm.*