MA500-1 Numerical Differential Equations

Problem Set 2

Q1. Let L be the differential operator: $L(u)(x) = u''(x)$. Show that L has *eigenfunctions* u^p and corresponding eigenvalues μ_p where

$$u^p = \sin(p\pi x) \text{ and } \mu_p = -p^2\pi^2,$$

for $p = 1, 2, 3, \dots$. Show how these eigenvalues relate to eigenvalues of the matrix that arises in the usual second-order finite-difference method for $L(u)(x) = f(x)$. (Hint: use a Taylor's series for the trigonometric term in the eigenvalues of the FDM).

Q2. Recall that in class we demonstrated that the $(N-1)\times(N-1)$ tridiagonal matrix with entries

$$a_{ij} = \begin{cases} \frac{1}{h^2} & |i-j| = 1 \\ -\frac{2}{h^2} & i = j \\ 0 & \text{otherwise} \end{cases}$$

where $h = 1/N$, has eigenvalues

$$\lambda_p = \frac{2}{h^2}\big(\cos(p\pi h) - 1\big),$$

with corresponding eigenvectors whose entries are

$$(u_p)_j = \sin(p\pi jh).$$

This is actually a special case of the following result: let A be the $(N-1)\times(N-1)$ tridiagonal matrix with entries

$$a_{ij} = \begin{cases} \gamma & j = i-1 \\ \alpha & j = i \\ \beta & j = i+1. \end{cases}$$

then the eigenvalues of A are

$$\lambda_p = \alpha + 2\sqrt{\beta\gamma}\cos(p\pi h), \tag{1a}$$

with corresponding eigenvectors whose entries are

$$(u_p)_j = (\gamma/\beta)^{j/2}\sin(p\pi jh), \tag{1b}$$

where (as usual) $h = 1/N$.

(a) Verify that the formulae in (1) are correct.

(b) Suppose that we apply the usual second-order difference scheme to the BVP

$$u''(x) + qu(x) = f(x) \quad \text{on } (0,1)$$

with $u(0) = u(1) = 1$, and q is a negative constant. Show that all the eigenvalues of the corresponding FDM matrix, A, are real, negative numbers. Give a bound for $\|A^{-1}\|_2$.

(c) Suppose that we apply a finite difference scheme to solving the BVP

$$u''(x) + pu'(x) = f(x) \quad \text{on } (0,1)$$

where p is a positive constant.

(i) If $u'(x_i)$ is approximated as

$$u'(x_i) = \frac{1}{2h}\big(-u(x_{i-1}) + u(x_{i+1})\big) + \mathcal{O}(h^2),$$

for what values of p will the FD matrix have real, negative eigenvalues? Give an estimate for $\lim_{h\to 0}\lambda_1$.

(ii) If $u'(x_i)$ is approximated as

$$u'(x_i) = \frac{1}{h}\big(-u(x_{i-1}) + u(x_i)\big) + \mathcal{O}(h),$$

for what values of p will the FD matrix have real, negative eigenvalues? Give an estimate for $\lim_{h\to 0}\lambda_1$.

Why does knowing λ_1 for A in Part (b) give us a bound for $\|A^{-1}\|_2$, but not for the corresponding matrices in Part (c)-(i) and (c)-(ii)?

Q3. Let L be the differential operator

$$L(u)(x) := u''(x) - u'(x) + u(x).$$

Show that L satisfies a *maximum principle*: If $Lu \geqslant 0$, for all $x \in [a,b]$, and $u(a) = \alpha \geqslant 0$, $u(b) = \beta \geqslant 0$, then $u \leqslant \max\{\alpha, \beta\}$.

Q4. Suppose that u solves $u''(x) + q(x)u(x) = f(x)$ on $(0,1)$ and $u(0) = u(1) = 0$, and that there is a negative constant q_0 such that $q(x) < q_0 < 0$ for all x. Show that $\|u\|_\infty \leqslant \|f\|_\infty/q_0$.

Q5. Suppose that u solves

$$\begin{aligned} u''(x) + qu(x) &= f(x) \quad \text{for } 0 < x < 1, \\ u(0) &= \alpha, u(1) = \beta, \end{aligned} \tag{2}$$

and that v solves

$$\begin{aligned} v''(x) + qv(x) &= g(x) \quad \text{for } 0 < x < 1, \\ v(0) &= v(1) = 0. \end{aligned}$$

Find g and y, a linear function in x such that $u(x) = v(x) + y(x)$.

Q6. Suppose that u solves

$$u''(x) = f(x) \quad \text{for } 0 < x < 1,$$

$$u'(0) = u(1) = 0.$$

Show that u is a non-decreasing function on $(0,1)$.

If the boundary conditions are instead

$$u(0) = u'(1) = 0,$$

show that u is non-increasing. (Hint: Fundamental theorem...).