MACSI One Day Graduate Course:
Numerical Solution to Differential Equations using Matlab

Part 3: Errors and Rates of Convergence

Niall Madden and Meghan Stephens
Department of Mathematics
National University of Ireland, Galway
Ollscoil na hÉireann, Gaillimh

3rd of April 2007.

Recall the differential equation:

Define the operator

$$L(u) := -u''(x) + r(x)u(x).$$

Then the general form of a BVP is: *find a function u defined on the interval* $[0, 1]$

$$L(u) = f(x) \text{ for } \quad 0 < x < 1, \qquad \text{and } u(0) = \alpha, u(1) = \beta.$$

Maximum Principle

Lets assume that $r(x) > 0$ for all $x \in [0, 1]$.

Lemma (Maximum Principle)

Suppose u is a function such that $Lu \geq 0$ *on* $(0, 1)$ *and* $u(0) \geq 0$, $u(1) \leq 0$. *Then* $u \geq 0$ *for all* $x \in [0, 1]$.

Proof:

Maximum Principle

This lemma is as useful as it is simple. For example,

Example

Let ϱ be such $r(x) \geq \varrho > 0$. Define $C = \max_{a \leq x \leq b} |f(x)|/\varrho$. Then $u(x) \leq C$.

Maximum Principle

Example

There is at most one solution to out differential equation.

Exercise

Suppose that we had the more general differential operator:

$$L_q(u) := -u''(x) + q(x)u'(x) + r(x).$$

Would this L_q also satisfy a maximum principle?

Maximum Principle

A *mesh function* is a set of real numbers $\{V_i\}_0^N$, where V_i is taken to mean the value of the function at $x = x_i$.

One may write $V(x)$, but with the understanding that V is defined only at the mesh points.

Let δ^2 be the difference operator:

$$\delta^2 V_i := \frac{1}{h^2}\big(V_{i-1} - 2V_i + V_{i+1}\big)$$

In analogy to the (continuous) differential operator, we define the **difference operator** L^h:

$$L^h(V)_i := -\delta^2 V_i + r(x_i)V_i \qquad \text{for } i = 1, \dots N-1.$$

Maximum Principle

Now our finite difference equation can be cast as: *Find the mesh function* $\{U_i\}_{i=0}^N$ *that satisfies*

$$L^h U_i = f(x_i) \quad \textit{for } i = 1, \dots N-1, \qquad \textit{and } U_0 = U_N = 0.$$

The problem now is to estimate the error.

A norm

First we need a norm.

The "max" norm $\|\cdot\|_\infty$ is defined as

$$\|u\|_\infty := \max_{0\le x\le 1} |u(x)| \quad \text{for any function that is continuous on } [0,1]$$

$$\|V\|_{\infty,\{x_i\}_0^N} := \max_{0\le i\le N} |V_i| \quad \text{for any mesh function on } \{x_i\}_{i=0}^N.$$

Usually, when it is clear what interval/mesh we are using, we simply write the norm as $\|\cdot\|_\infty$, or even just $\|\cdot\|$.

Another Maximum Principle

Lemma (Discrete Maximum Principle)

Suppose that $\{V_i\}_{i=0}^N$ *is a mesh function such that*

$$L^h V_i \geq 0 \text{ on } x_1, \dots x_{N-1},$$

and

$$V_0 \geq 0, V_N \geq 0.$$

Then $V_i \geq 0$ *for* $i = 0, \dots N$.

Exercise

Proving this lemma is a nice exercise. Use an argument similar to the one which previous Max Prin.

Another Maximum Principle

An simple consequence of this lemma is

Let $\{V_i\}_{i=0}^N$ be any mesh function with $V_0 = V_N = 0$. Then

$$|V_i| \leq \varrho^{-1} \|L^h V_i\|_\infty$$

Error Estimates

We can now use the above results to show that

Theorem

Suppose that $u(x)$ *is the solution to the problem:*

$$Lu(x) = f(x), \qquad u(0) = u(1) = 0$$

and $\|u^{(iv)}(x)\|_\infty \leq M$. *Let* U *be the mesh function that solves*

$$L^h U_i = f(x_i) \quad \text{for } i = 1, 2, \dots, N-1, \qquad U_0 = U_N = 1.$$

Then

$$\|u - U\| := \max_k |u(x_k) - U_k| \leq \frac{h^2}{12} \frac{M}{\varrho}$$

Error Estimates

It is usual to restate this results as little less formally:

There are constants C and γ that do not depend on N such that

$$\|u - U\| \leq CN^{-\gamma}.$$

That is:

- The rate of convergence is of the method is γ. So in our case, $\gamma = 2$ and we say the method is second order.
- The constant of convergence is C. It depends in the data of the differential equations: $r(x)$, $f(x)$, the boundary conditions, and on the derivatives of $u(x)$.