Classification of cocyclic Hadamard matrices of order ≤ 28

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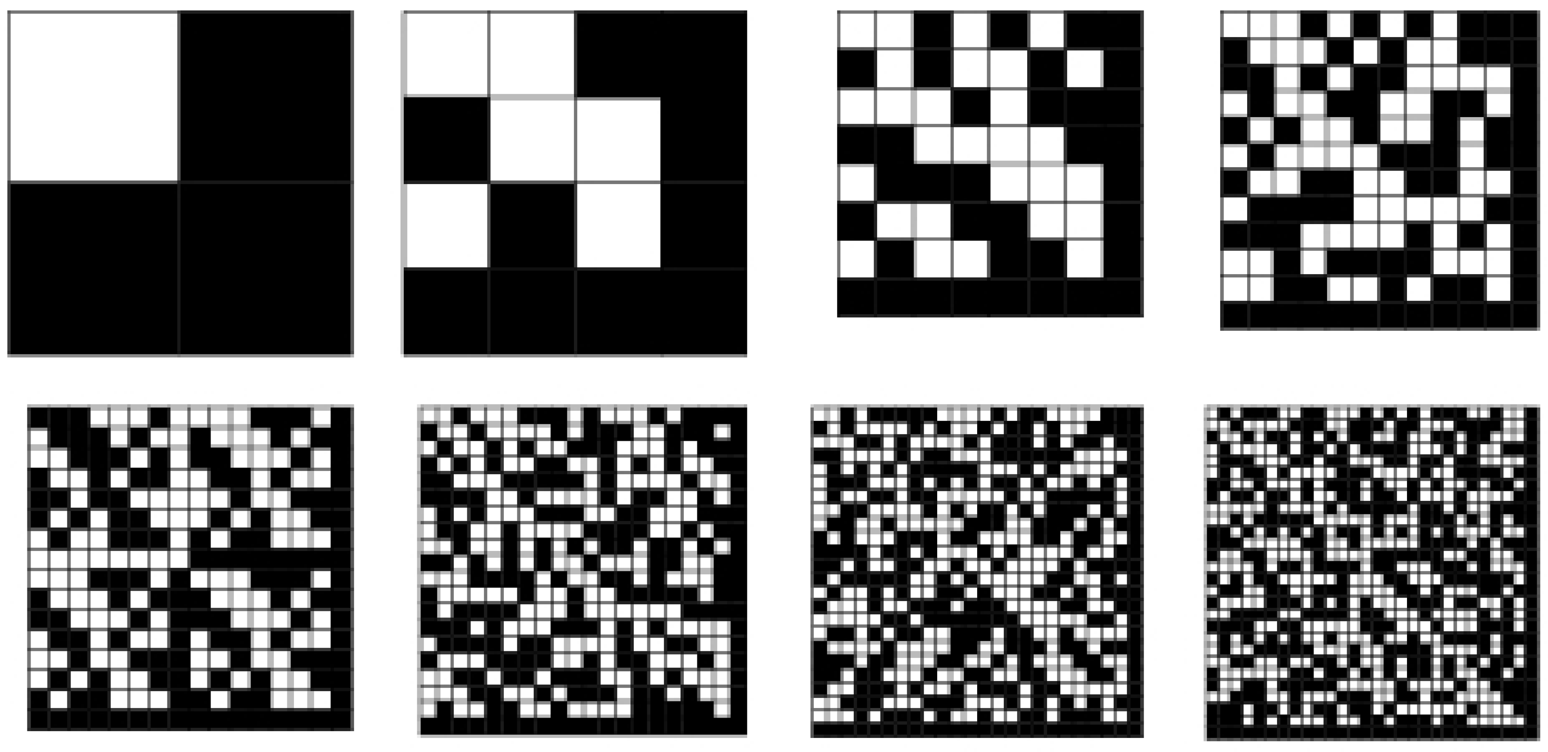
Introduction

- A ± 1 matrix of order n which satisfies the equation $HH^T = nI_n$ is called a Hadamard matrix
- Hadamard matrices are used in coding theory, statistics, combinatorial design theory, and many other fields of mathematics
- It is conjectured that there exists a Hadamard matrix of order $4n$ for all $n \in N$
- Cocyclic development brings some of the tools of group theory and cohomology theory to bear on unsolved problems concerning Hadamard matrices

Hadamard matrices of small order



Anallagmatic pavements were introduced by Sylvester. It was from diagrams such as these these that he developed the first family of Hadamard matrices. In the above pavements, the tiles of any pair of columns match in half of their positions and are opposite in the remaining half. This definition guarantees orthogonality if we replace black and white squares with 1 and -1. In fact, it is equivalent to the Hadamard property.

Group development of matrices

A matrix, M, with entries in a set C is developed over a group G, if there exists a function $\varphi : G \to C$ such that

$$M = (\varphi(gh))_{g,h \in G}$$

where the rows and columns of M are labelled with elements of G.
For example, a Hadamard matrix can be group developed over C_4. Let $\varphi : C_4 \to \langle -1 \rangle$ be given by $\varphi\left(c^2\right) = -1$, $\varphi(1) = \varphi(c) = \varphi\left(c^3\right) = 1$. Then

$$\varphi\left(\begin{bmatrix} 1 & c & c^2 & c^3 \\ c & c^2 & c^3 & 1 \\ c^2 & c^3 & 1 & c \\ c^3 & 1 & c & c^2 \end{bmatrix}\right) = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

Cocyclic development is an extension of group development to functions of two variables. This is necessary because group developed matrices must have constant row and column sums, but Hadamard matrices can only have this property if they have square order.

Group development is also intimately related to the concept of regular actions. In fact, a matrix, M, is group developed over a group, G of order n, if and only if there exist permutation matrix representations of G which act regularly on both the rows and the columns of M.

Cocyclic development

Let G be a finite group, and C a finitely generated Abelian group. A 2-cocycle is a map $\varphi : G \times G \mapsto C$ which satisfies the equation

$$\varphi(g,h)\,\varphi(gh,k) = \varphi(g,hk)\,\varphi(h,k) \quad \forall g,h,k \in G$$

Then M is cocyclic developed with cocycle φ if

$$M = (\varphi(g,h))_{g,h \in G}$$

It is conjectured that cocyclic developed Hadamard matrices exist at all possible orders. We can identify cocyclic Hadamard matrices by special properties of their automorphism groups.

We define an automorphism of a Hadamard matrix, H, to be an ordered pair of signed permutation matrices, (P,Q), such that

$$PHQ^T = H$$

E_H is the expanded design of the Hadamard matrix, H. It is the final tool necessary to fully describe the cocyclic development of a matrix.

$$E_H = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes H$$

Let H be a Hadamard matrix of order n, and $\varphi : G \times G \to \langle -1 \rangle$ a cocycle such that $Aut(H)$ contains a subgroup, J, isomorphic to $E(\varphi)$, containing a specific central involution. Then H is cocyclic developed with cocycle φ if and only if there exists a regular action of J, on the expanded design E_H. The automorphism group can contain many regular subgroups, in which case a Hadamard matrix can be cocyclic developed over several non-isomorphic groups.

Results

- All Hadamard matrices are known only for orders ≤ 28
- The table below outlines the main results of the classification
- Where a fraction occurs, this is the proportion of all objects at that order with the required property

Order	Cocyclic	Indexing Groups	Extension Groups
2	1	1	2
4	1	2	3 / 5
8	1	3 / 5	9 / 14
12	1	3 / 5	3 / 15
16	5	13 / 14	45 / 51
20	3	2 / 5	3 / 14
24	18 / 60	6 / 15	15 / 52
28	6 / 487	2 / 4	2 / 13

- All Hadamard matrices of order ≤ 20 are cocyclic
- The smallest order for which the existence of cocyclic Hadamard matrices is unknown is 188

Conclusion

- Cocyclic development can be used to classify Hadamard matrices
- Many different constructions of Hadamard matrices are cocyclic
- It may be possible to extend cocyclic development to prove the existence of Hadamard matrices of order $4n$ for all $n \in N$
- Such a result would settle the Hadamard conjecture

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