REALIZING FUSION SYSTEMS INSIDE FINITE GROUPS

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ABSTRACT. We show that every (not necessarily saturated) fusion system can be realized as a full subcategory of the fusion system of a finite group. This result extends our previous work [5] and complements the related result [4] by Leary and Stancu.

1. Statements of the results

Fix a prime $p$. Let $G$ be a finite group, and let $S$ be a $p$-subgroup of $G$. We denote by $\mathcal{F}_S(G)$ the category whose objects are the subgroups of $S$ and such that for $P, Q \leq S$ we have

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) = \{ \varphi: P \to Q \mid \exists x \in G \text{ s.t. } \varphi(u) = xu^{-1} \text{ for } u \in P \},$$

where composition of morphisms is composition of functions.

The category $\mathcal{F}_S(G)$ above is a fusion system on $S$. If $S$ is a Sylow $p$-subgroup of $G$, then $\mathcal{F}_S(G)$ is saturated, but not all saturated fusion systems are of this form. Those saturated fusion systems $\mathcal{F}$ such that $\mathcal{F} \neq \mathcal{F}_S(G)$ for any finite group $G$ having $S$ as a Sylow $p$-subgroup are called exotic fusion systems. We refer the reader to [1] for precise definitions and a general introduction to the subject. In [5] we showed that every saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$ is of the form $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group $G$ having $S$ as a subgroup. The point here is that we are not requiring that $S$ is a Sylow $p$-subgroup of $G$. In this short note, we observe that this result holds even when $\mathcal{F}$ is not saturated.

**Theorem 1.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. Then there is a finite group $G$ having $S$ as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.

Thus fusion systems are precisely those categories of the form $\mathcal{F}_S(G)$ for some finite group $G$ and a $p$-subgroup $S$ of $G$. Leary and Stancu [4] showed that every fusion system is of the form $\mathcal{F}_S(G)$ where $G$ is a (possibly infinite) group having $S$ as a Sylow $p$-subgroup, in the sense that every finite $p$-subgroup of $G$ is conjugate to a subgroup of $S$. Here $\mathcal{F}_S(G)$ is defined exactly the same way as when $G$ is a finite group. Leary and Stancu’s construction uses HNN extensions. As in [5], the proof of Theorem 1 uses a certain $S$-$S$-biset associated to the fusion system $\mathcal{F}$; though we use a slightly different one here. We keep the notations of [5].

**Definition 2.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. A left semicharacteristic biset for $\mathcal{F}$ is a finite $S$-$S$-biset $X$ satisfying the following properties:

1. $X$ is $\mathcal{F}$-generated, i.e., every transitive subbiset of $X$ is of the form $S \times_{(Q, \varphi)} S$ for some $Q \leq S$ and some $\varphi \in \text{Hom}_{\mathcal{F}}(Q, S)$.
2. $X$ is left $\mathcal{F}$-stable, i.e., $QX \cong \varphi X$ as $Q$-$S$-bisets for every $Q \leq S$ and every $\varphi \in \text{Hom}_{\mathcal{F}}(Q, S)$. 

1
A right semicharacteristic biset is defined analogously with right $\mathcal{F}$-stability instead of left $\mathcal{F}$-stability; a semicharacteristic biset is a biset which is both left and right semicharacteristic. When the fusion system is saturated, semicharacteristic bisets are parametrized by Gelvin and Reeh [3] using a result of Reeh [7], and left semicharacteristic bisets can be parametrized analogously. A left characteristic biset is a left semicharacteristic biset $X$ such that $|X|/|S| \neq 0 \pmod{p}$. Broto–Levi–Oliver [2, Proposition 5.5] showed that every saturated fusion system has a left characteristic biset $X$. In [5], we used this biset $X$ to construct the finite group $G$ in Theorem 1 when $\mathcal{F}$ is saturated. Here we show that every fusion system has a certain left semicharacteristic biset $X$ with an additional property which falls short of making $X$ left characteristic, but which still ensures that the proof in [5] carries over.

**Proposition 3.** Every fusion system $\mathcal{F}$ on a finite $p$-group $S$ has a left semicharacteristic biset $X$ containing $S \times_{(S,\text{id})} S$.

We are going to prove Proposition 3 and Theorem 1 in the next section.

**Remark 4.** In [6, Proposition 3.1], a semicharacteristic biset containing $S \times_{(S,\text{id})} S$ is used for a saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$. Thus Proposition 3 tells us that [6, Proposition 3.1] holds for an arbitrary fusion system $\mathcal{F}$.

### 2. Semicharacteristic bisets for fusion systems

Let $G$ be a finite group. A virtual $G$-set with rational coefficients is an element of the rational Burnside ring $\mathbb{Q} \otimes_{\mathbb{Z}} B(G)$, i.e., a formal sum

$$\sum_{H} c_H G/H$$

where $H$ runs over a set of representatives of conjugacy classes of subgroups of $G$ and $c_H \in \mathbb{Q}$. If the coefficients of a virtual $G$-set are all nonnegative integers, then it is simply a (isomorphism class of) finite $G$-set.

The key step of the proof of Proposition 3 is the following lemma, which says roughly that every virtual $S$-set with rational coefficients can be stabilized (with respect to a given fusion system $\mathcal{F}$) by adding a virtual $S$-set with nonnegative rational coefficients.

**Lemma 5 (cf. [2, Lemma 5.4]).** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. Let $\mathcal{H}$ be a collection of subgroups of $S$ which is closed under $\mathcal{F}$-conjugation and taking subgroups. Let $X_0$ be a virtual $S$-set with rational coefficients such that $|X_0^P| = |P|$, for all $P$, $P' \leq S$ with $P, P' \notin \mathcal{H}$ which are $\mathcal{F}$-conjugate. Then there is a virtual $S$-set $X$ with rational coefficients such that $|X^P| = |P|$ for all $P, P' \leq S$ which are $\mathcal{F}$-conjugate, $|X_{0}^P| = |X_{0}^P|$ for all $P \leq S$ with $P \notin \mathcal{H}$, and $X - X_0$ is a virtual $S$-set with nonnegative rational coefficients.

**Proof.** Consider an $\mathcal{F}$-conjugacy class $\mathcal{P}$ of subgroups of $S$ in $\mathcal{H}$ which are maximal among such subgroups. Choose $P \in \mathcal{P}$ such that $|X_0^P| \geq |X_0^{P'}|$ for all $P' \in \mathcal{P}$. Set

$$X_1 = X_0 + \sum_{P'} \frac{|X_0^{P'}| - |X_0^P|}{|N_S(P')/P'|} S/P',$$

where $P'$ runs over a set of representatives of the subgroups in $\mathcal{P}$ up to $S$-conjugacy. Then for any $P' \in \mathcal{P}$, we have $|X_1^{P'}| = |X_0^{P'}| = |X_0^P|$. Note that $|X_1^P| = |X_0^P|$ for
all $P \leq S$ with $P \notin \mathcal{H}$, and hence $|X_P^P| = |X_P'|$ for all $P, P' \leq S$ with $P, P' \notin \mathcal{H} \setminus \mathcal{P}$ which are $\mathcal{F}$-conjugate. Also, $X_1 - X_0$ is a virtual $S$-set with nonnegative rational coefficients. So by repeating this process we get a virtual $S$-set $X$ with the desired properties. □

Comparing the above lemma to [2, Lemma 5.4], we see that here the lack of saturation is compensated for by allowing rational coefficients.

**Proof of Proposition 3.** Let
$$Y_0 = \sum_{\alpha \in \text{Out}_F(S)} S \times_{(S, \alpha)} S.$$ 

Then $Y_0$ satisfies the assumption of Lemma 5 with respect to the product fusion system $\mathcal{F} \times \mathcal{F}_S(S)$ on $S \times S$ and $\mathcal{H} = \{ \Delta(P, \varphi) \mid P < S, \varphi \in \text{Hom}_F(P, S) \}$. (See [1, Definition I.6.5, Theorem I.6.6] for the definition and properties of the product fusion system.) Thus Lemma 5 implies that there is a virtual $S$-set $Y$ with nonnegative rational coefficients which is $\mathcal{F}$-generated and left $\mathcal{F}$-stable and which contains $S \times (S, \text{id}) S$. Let $m$ be a large enough positive integer such that $X = mY$ is a (finite) $S$-set (with nonnegative integer coefficients). Then $X$ is a left semicharacteristic biset for $\mathcal{F}$ containing $S \times (S, \text{id}) S$. □

**Proof of Theorem 1.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$ and let $X$ be a left semicharacteristic biset for $\mathcal{F}$ containing $S \times (S, \text{id}) S$. Let $G$ be the group of automorphisms of $X$ viewed as a right $S$-set, i.e., the group of bijections $f : X \to X$ such that $f(xs) = f(x)s$ for all $x \in X$ and $s \in S$. Then $S$ embeds into $G$ via
$$S \to G, \quad s \mapsto (x \mapsto sx).$$

The proof of [5, Theorem 6] applies verbatim to this situation. Thus we have $\mathcal{F} = \mathcal{F}_S(G)$. □

**References**


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