Outline

1. Modular representation theory
   - Modules and vertices
   - Blocks and defect groups
   - Representation types of block algebras

2. Fusion systems—from ‘local’ to ‘global’
   - Fusion systems of finite groups
   - Saturated fusion systems
   - $p$-local finite groups and characteristic idempotents

3. My work
   - Alperin’s weight conjecture
   - Control of fusion and transfer
   - Relations between characteristic idempotents
   - Realization and the exoticity index
Modular representation theory

$G$ finite group, $k$ algebraically closed field

**Theorem (Maschke)**

$kG$ is semisimple iff \( \text{char } k \nmid |G| \).

Modular rep theory = study of \( \text{mod}(kG) \) when \( \text{char } k = p \mid |G| \).

Not all $kG$-modules are projective $\Rightarrow$ measure the failure by vertices.

**Definition**

Let $M$ be an indecomposable $kG$-module; $Q \leq G$.

1. $M$ is relatively $Q$-projective if $M \mid \text{Ind}_Q^G(N)$ for some $kQ$-module $N$.
2. A vertex of $M$ is a minimal subgroup $Q$ of $G$ such that $M$ is relatively $Q$-projective.

- Vertices are $p$-subgroups of $G$, unique up to $G$-conjugacy.
- $M$ is projective iff $M$ has vertex 1.
Indecomposable factors of the algebra $kG$ are called the blocks of $kG$:

$$kG = B_1 \times \cdots \times B_r$$

$$1 = e_1 + \cdots + e_r$$

If $M$ is a $kG$-module, then

$$M = e_1 M \oplus \cdots \oplus e_r M$$

In particular, if $M$ is indecomposable, then $M = e_i M$ for some (unique) $i$ and $e_j M = 0$ for all $j \neq i$. In this case, we say $B_i$ contains $M$. The unique block of $kG$ containing $k$ is called the principal block.

**Definition**

A defect group of a block $B$ is a minimal $P \leq G$ such that every indecomposable $B$-module is relatively $P$-projective.
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**Definition**

A defect group of a block $B$ is a minimal $P \leq G$ such that every indecomposable $B$-module is relatively $P$-projective. [Or, minimal $P \leq G$ s.t. res: $D^b(modB) \to D^b(modkP)$ is faithful.]

- Defect groups are $p$-subgroups of $G$, unique up to $G$-conjugacy.
- $B \cong M_n(k)$ for some $n$ iff $B$ has defect group 1.
- For the principal block: Defect groups $\equiv$ Sylow $p$-subgroups
Theorem (Bondarenko-Drozd)

Let $B$ be a block of $kG$ with defect group $P$.

1. $B$ has finite representation type iff $P$ is cyclic.
2. $B$ has tame representation type iff $p = 2$ and $P$ is dihedral, semidihedral, or generalized quaternion.
3. $B$ has wild representation type in all other cases.

How to study block algebras

“Local-Global Principle”

1. From ‘local’: information of $p$-subgroups of $G$ and conjugation maps between them
2. To ‘global’: information of the whole group $G$

Local structure of block algebras can be described using fusion systems.
Fusion systems of finite groups

Let $p$ be a fixed prime, $G$ a finite group, $P \in \text{Syl}_p(G)$.

**Definition**

The **fusion system** of $G$ on $P$ is the category $\mathcal{F} = \mathcal{F}_P(G)$:

- objects: $Q \leq P$
- morphisms: $\text{Hom}_\mathcal{F}(Q, R) = \text{Hom}_G(Q, R)$

**Theorem (Frobenius)**

$G$ is $p$-nilpotent, i.e. $G = K \rtimes P$ for some $K$ iff $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

**Theorem (Cartan-Eilenberg)**

$$H^*(BG, \mathbb{F}_p) \cong \varprojlim \mathcal{F}_P(G) H^*(-, \mathbb{F}_p)$$
Saturated fusion systems

Definition (Puig)

A saturated fusion system on a finite $p$-group $P$ is a category $\mathcal{F}$:

- objects: $Q \leq P$
- morphisms: $\text{Hom}_P(Q, R) \subseteq \text{Hom}_\mathcal{F}(Q, R) \subseteq \text{Inj}(Q, R)$

such that “morphisms behave as if they were $G$-conjugation maps for some $G$ with $P \in \text{Syl}_p(G)$”.

- $\mathcal{F}_P(G)$ for a finite group $G$ with $P \in \text{Syl}_p(G)$.
- $\mathcal{F}_P(B)$ for a block $B$ with defect group $P$. (Alperin-Broué-Puig)
- $\exists \mathcal{F} \neq \mathcal{F}_P(G)$ for any $G$ with $P \in \text{Syl}_p(G)$ (exotic fusion systems)

Theorem (Puig)

Let $B$ be a block of $kG$ with defect group $P$. If $B$ is nilpotent, i.e. $\mathcal{F}_P(B) = \mathcal{F}_P(P)$, then $B$ is Morita equivalent to $kP$. 
**Conjecture**

For every sfs $\mathcal{F}$ on $P$, there is a finite category $\mathcal{L}$ such that

$$H^*(|\mathcal{L}|_p^\wedge, \mathbb{F}_p) \cong \lim_{\mathcal{F}} H^*(-, \mathbb{F}_p)$$

$(P, \mathcal{F}, \mathcal{L})$ a $p$-local finite group with classifying space $|\mathcal{L}|_p^\wedge$

**Theorem (Broto-Levi-Oliver)**

When $\mathcal{F} = \mathcal{F}_P(G)$ where $P \in \text{Syl}_p(G)$, $\mathcal{L}$ exists

**Theorem (Broto-Levi-Oliver; Ragnarsson)**

Every sfs $\mathcal{F}$ on $P$ uniquely determines an idempotent $\omega_\mathcal{F}$ in the double Burnside ring $A(P, P) \otimes_{\mathbb{Z}} \mathbb{Z}(p)$, which corresponds to a summand $B\mathcal{F}_+$ of $\Sigma^\infty BP_+$ such that $B\mathcal{F}_+ \simeq \Sigma^\infty |\mathcal{L}|_p^\wedge$ (if $\mathcal{L}$ exists).

$\omega_\mathcal{F}$ characteristic idempotent of $\mathcal{F}$ (e.g. $\mathcal{F} = \mathcal{F}_P(G) \rightsquigarrow \omega_\mathcal{F} \rightsquigarrow G$)
Alperin’s Weight Conjecture

Let $B$ a block of $kG$, $P \neq 1$ defect group of $B$, $\mathcal{F} = \mathcal{F}_P(B)$. Then

$$\ell(B) = \sum_{[Q]_G} \# \text{ proj simples of } kN_G(Q)/Q \text{ assoc. with } B$$

$$= \sum_{[Q]_{\mathcal{F}^c}} z(k_{\alpha(Q)} \text{Out}_\mathcal{F}(Q))$$

where $\alpha(Q) \in H^2(\text{Out}_\mathcal{F}(Q), k^\times)$, $Q \in \text{Ob}(\mathcal{F}^c)$ (Külshammer-Puig class).

Gluing Problem (Linckelmann)

*Is there $\alpha \in H^2(\mathcal{F}^c, k^\times)$ which restricts to KP classes for all $Q \in \text{Ob}(\mathcal{F}^c)$?*

If so,

Theorem (Linckelmann)

$$AWC \iff \ell(B) = \ell(k_\alpha \mathcal{F}^c(B)) \text{ where } k_\alpha \mathcal{F}^c(B) = e_z(k_\alpha \mathcal{F}^c)e_z.$$
Theorem (Linckelmann-P, 2008)

Let $B$ be a block with defect $d$ for which GP has a solution $\alpha$.

1. $k_\alpha \overline{F}^c(B)$ is a quasi-hereditary algebra.
2. Every standard module is projective.
3. $\text{gldim}(k_\alpha \overline{F}^c(B)) \leq 2(d - 1)$.

Theorem (P, 2008)

Let $B$ be the principal $2$-block of $GL_2(q)$, $q$ odd prime power. Then

$$B \rightarrow S_2(q) \rightarrow k_0 \overline{F}^c(B) \quad \text{up to Morita equivalence},$$

which gives a 1-1 correspondence between simples and weights for $B$.

Theorem (P, 2009)

1. GP has a unique solution for tame blocks.
2. GP for the principal $p$-block of $\text{PSL}_3(p)$ ($p$ odd) has a unique solution if $p \not\equiv 1 \mod 3$; three solutions if $p \equiv 1 \mod 3$. 
Control of fusion and transfer

**Theorem (Thompson; Díaz-Glesser-Mazza-P, 2009)**

Let $\mathcal{F}$ be a sfs on a finite $p$-group $P$. Suppose that $p$ is odd or that $\mathcal{F}$ is $S_4$-free. If $C_{\mathcal{F}}(Z(P)) = N_{\mathcal{F}}(J(P)) = \mathcal{F}_P(P)$, then $\mathcal{F} = \mathcal{F}_P(P)$.

Proof by reduction to group case

**Theorem (Tate; Díaz-Glesser-P-Stancu, 2010)**

Let $\mathcal{F}$ be a sfs on a finite $p$-group $P$. Then

$$\text{res}: \lim_{\mathcal{F}} H^1(-, \mathbb{F}_p) \sim H^1(P, \mathbb{F}_p) \iff \mathcal{F} = \mathcal{F}_P(P).$$

Proof by using characteristic idempotents $\omega_{\mathcal{F}}$
Relations between characteristic idempotents

Let $G$ be a finite group, $P, L \leq G$, $G = PL$, $N = P \cap L$. Then

$L/N \cong G/P$ \quad a bijection of coset spaces,

or equivalently

$P \times_N L \cong G$ \quad as $(P, N)$-bisets.

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Theorem (P-Ragnarsson-Stancu)

Let $(\mathcal{F}, P) \geq (\mathcal{H}, P), (\mathcal{K}, N)$. Suppose $p_{\mathcal{F}N} = \mathcal{H}\mathcal{K}$ and $\mathcal{K}$ is normal in $\mathcal{F}$. Then

$\omega_{\mathcal{H}} \times_N \omega_{\mathcal{K}} = \omega_{\mathcal{F}|N}$
The exoticity index of a fusion system

$G$ finite group, $P$ a $p$-subgroup of $G$
Can define $\mathcal{F}_P(G)$ even when $P$ is not a Sylow $p$-subgroup of $G$.

**Theorem (P, 2010)**

Let $P$ be a finite $p$-group. For any sfs $\mathcal{F}$ on $P$, there is a finite group $G$ with $P \leq G$ such that $\mathcal{F} = \mathcal{F}_P(G)$.

Proof by construction using characteristic idempotent $\omega_{\mathcal{F}}$.

**Definition**

Let $\mathcal{F}$ be a sfs on a finite $p$-group $P$. Define the **exoticity index** $e(\mathcal{F})$ of $\mathcal{F}$ to be

$$\min\{\log_p |P_0 : P| \mid P \leq P_0 \in \text{Syl}_p(G), \mathcal{F} = \mathcal{F}_P(G)\}$$

- $e(\mathcal{F}) \in \mathbb{Z}_{\geq 0}$
- $\mathcal{F}$ exotic $\iff e(\mathcal{F}) > 0$
**Theorem (Ruiz-Viruel)**

Let $p$ be odd and $\mathcal{F}$ a sfs on $P \in \text{Syl}_p(\text{PSL}_3(p))$ s.t. all $V_i \leq P$ of index $p$ are $\mathcal{F}$-radical. Then $\mathcal{F}$ is one of the following:

| $p$ | $\text{Out}_\mathcal{F}(P)$ | $|V_i^\mathcal{F}|$ | $r_i$ | Group          |
|-----|--------------------------|-------------------|-------|----------------|
| 3   | $D_8$                    | 2, 2              | 2, 2  | $2F_4(2)'$     |
| 3   | $SD_{16}$                | 4                 | 2     | $J_4$          |
| 5   | $4S_4$                   | 6                 | 4     | $Th$           |
| 7   | $D_{16} \times 3$       | 4, 4              | 2, 2  |                |
| 7   | $6^2 : 2$                | 6, 2              | 2, 6  |                |
| 7   | $SD_{32} \times 3$      | 8                 | 2     |                |

where $\text{Aut}_\mathcal{F}(V_i) \cong \text{SL}_2(p) \rtimes \mathbb{Z}/r_i$
Theorem (Ruiz-Viruel; P, 2010)

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| 3   | $D_8$           | 2, 2            | 2, 2| $2F_4(2)'$     |
| 3   | $SD_{16}$       | 4               | 2   | $J_4$          |
| 5   | $4S_4$          | 6               | 4   | $Th$           |
| 7   | $D_{16} \times 3$ | 4, 4          | 2, 2| $\leq 425744$ |
| 7   | $6^2:2$         | 6, 2            | 2, 6| $\leq 638620$ |
| 7   | $SD_{32} \times 3$ | 8         | 2   | $\leq 851496$ |

where $\text{Aut}_\mathcal{F}(V_i) \cong \text{SL}_2(p) \rtimes \mathbb{Z}/r_i$. 