Mathematical and **Logic**al Aspects of Computing (CS304/CS310)

Lecture 13: Natural Deduction

Tuesday, the 22^{nd} of October 2013

Natural Deduction

In the previous, we introduced the notion of *Deductive systems* which formalise mathematical reasoning. It involves:

- A collection of rules: eleven in the system we'll study,
- Careful proofs of certain propositions that follow from these rules.

In this class, we largely follow the presentation in Kelly "Essence of Logic".

. . . .

Rule 1: $\wedge$-introduction ("$\wedge$-I")

$$\frac{A \qquad B}{A \wedge B}.$$

Example:

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Rule 2: $\wedge$-elimination ("$\wedge$-E")

$$\frac{A \wedge B}{A}.$$

Example:

Rule 3: $\wedge$-elimination ("$\wedge$-E")

$$\frac{A \wedge B}{B}.$$

Rule 4: $\vee$-introduction ("$\vee$-I")

$$\frac{A}{A \vee B}.$$

Example:

Rule 5: $\vee$-introduction; ("$\vee$-I")

$$\frac{B}{A \vee B}.$$

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Rule 6: $\vee$-elimination ("$\vee$-E")

$$\dfrac{A \vee B \qquad \begin{matrix} \cancel{A} \\ \vdots \\ C \end{matrix} \qquad \begin{matrix} \cancel{B} \\ \vdots \\ C \end{matrix}}{C}$$

Example:

Rule 7: $\to$-introduction ("$\to$-I")

$$\dfrac{\begin{matrix} \cancel{A} \\ \vdots \\ C \end{matrix}}{A \to C}$$

Example:

Rule 8: $\to$-elimination ("$\to$-E"; "modus ponens")

$$\dfrac{A \qquad A \to C}{C}$$

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Rule 9: (falsum)

$$\frac{\bot}{C}$$

Rule 10: RAA (Reductio Ad Absurdum)

$$\begin{array}{c} \neg A \\ \vdots \\ \dfrac{\bot}{A} \end{array}$$

Rule 11: Id (Identity)

$$\frac{A}{A}$$

Example:

Natural Deduction

Equipped with these rules, we can now deduce some theorems. They have the form

$$\{A_1, A_2, \ldots, A_n\} \vdash C,$$

meaning that the conclusion C can be derived from the assumptions A_1, A_2, ..., A_n. By $\vdash C$ we mean that C can be derived without any assumptions

Theorem

$$\{A, B\} \vdash B \wedge A.$$

(7/11) Natural Deduction

Our next example is very closely related to the above one, but does not require any assumptions:

Theorem

$$\vdash (A \wedge B) \rightarrow (B \wedge A).$$

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Example: prove that

$$\vdash A \to \big(B \to (B \land A)\big).$$

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When we have a formula of the type $X \rightarrow Y$, it is often useful to make X an assumption with a view to discharging it later with Rule 7.
Example: prove

$$\{B\} \vdash A \rightarrow B.$$

Natural Deduction

It is often useful to note that $\neg A$ is equivalent to $A \to \perp$. Example: prove

$$\{A, \neg A\} \vdash \perp$$

. . . .

We can now use the above example to prove, e.g., that

$$\{\neg A\} \vdash A \to B.$$

(11/11) Natural Deduction

We will now prove the transposition rule treated as a homework from Friday: To show that

$$\vdash (\neg B \to \neg A) \to (A \to B).$$