Lecture 14: Introduction to Predicate Logic

Mathematical and Logical Aspects of Computing (CS304/CS310)

Friday, $25^{\rm th}$ and Tuesday, $29^{\rm th}$ of October 2013

 From Propositional to Predicate Logic

In Propositional Logic, we cannot manage to express the following simple syllogism:

> *All men are mortal*
> *Socrates is a man*
> *Therefore, Socrates is mortal.*

Thus, a new form of logic is needed: Predicate Logic.

 Predicate Logic

Predicates

"The subject of the sentence, as its name suggests, is generally what the sentence is about–its topic. The predicate is what is said about the subject."

"Predicate: something that is affirmed or denied of the subject in a proposition in logic."

So (informally) a predicate is a statement involving a verb (usually "is") that describes a property of a element or a set, or relations between elements. **Example:**

So, a predicate with variables is *not* a proposition. But it can be transformed into one. **Examples:**.
There are two ways to do this

1. Specify values for the variables;
2. Quantify the variables.

Examples

Quantifiers

- the *Universal* quantifier – "FOR ALL" $\forall$
- the *Existential* quantifier – "THERE EXISTS" $\exists$.

$$\exists x \bullet P(x) \equiv \neg \forall x \bullet \big(\neg P(x)\big).$$

Examples:

(5/18) Terms

A **term** is an expression used to name something. It can take the form

(a) **Constant symbols**;

(b) **Variables**;

Also:

(c) Descriptions;

(d) Complex mathematical expressions.

Examples:

Variables that occur in a (mathematical) expressions can be designated as **free** or **bound**. "Free" means we can read it as a name, for example substituting an arbitrary value for it. Otherwise it is "bound".

Examples: (sums, integrals; limits).

(6/18) Predicates

A **predicate** is a statement that may be true of false, depending on the values of its variables. That is, *a predicate maps elements of one or more sets to the truth values* $\{T, F\}$. **Examples:**

We typically represent a predicate either as an uppercase letter or, as typical in mathematics, with *infix notation*. **Examples:**

Terminology: nullary, unary, binary, n-ary

(7/18) Relations

Unary and binary predicates are the most important/interesting. In particular, binary predicates can be used to define a relation.
In general, binary relations can be defined by either listing all 2-tuples, or by expressing it as a binary predicate. **Example:**

The notation xRy is often used instead if $R(x, y)$ (recall "infix"/"prefix")

Universality and existence quantifiers

"for all" $\forall$ "there exists" $\exists$.

Of course, verifying that $\forall x P(x)$ is false requires finding any x where $P(x)$ is false. **Example:**

Hence $\forall x P(x)$ being false is equivalent to $\exists x \, \mathrm{not} \, P(x)$ being true

 Universality and existence quantifiers

Establishing that $\exists x Q(x)$ is **true** requires that we find **some** x where $Q(x)$ is true. **Example:**

And showing that $\exists x Q(x)$ is false means substituting **each** x into $Q(x)$ will **always** yield that $Q(x)$ is false. ($\forall x \, \mathrm{not} \, Q(x)$ is true)

These last examples show that we can easily relate these two quantifiers:
Example Which of the following statements are correct?

1. "Everyone likes reading or likes watching TV" is equivalent to "There is noone who likes both reading and watching TV".
2. "Everyone who likes reading, does not like watching TV" is equivalent to "There is noone who likes both reading and watching TV".

Answer:

Quantifier	True if ...	False if ...
$\forall x P(x)$	$P(x)$ is true for *every* x	There is some x for which $P(x)$ is *false*.
$\exists x P(x)$	$P(x)$ is true for *at least one* x	There is *no* x for which $P(x)$ is *true*.

Example:

. . . .

When a quantifier is applied to an occurrence of variable, or is a specific value is given to that variable, then that occurrence is *bound*. Otherwise it is *free*.

. . . .

The **scope** of a quantifier is the set of occurrences to which it applied.
Example:

We'll return to this next week when we look at *"parse trees"*.

Universe of discourse

We think of a predicate as a function that maps elements from some set to the truth values $\{F, T\}$. The domain of this predicate is often called the *universe of discourse*.

Examples:

Expressing quantifiers without $\forall$ or $\exists$

If the universe of discourse of a predicate is a finite set $X = \{x_1, x_2, \ldots x_n\}$ then we can express the quantifiers $\forall x \in X$ and $\exists x \in X$ by using $\wedge$ and $\vee$. **Here's how:**

de Morgan, yet again...

In the first part of this course, we studied numerous equivalences in propositional logic. The most fundamental of these are *de Morgan's laws:*

de Morgan

$$\text{(i) } \neg\,(a \wedge b) \;\equiv\; \neg a \vee \neg b \qquad\qquad \text{(ii) } \neg\,(a \vee b) \;\equiv\; \neg a \wedge \neg b.$$

For expressions involving quantifiers, we now know the most fundamental equivalences are:

$$\neg\forall x P(x) \;\equiv\; \exists x \neg P(x). \qquad \neg\exists x P(x) \;\equiv\; \forall x \neg P(x).$$

Of course: these four equivalences are all the same! **Example:**

 Summary

Summary. We have learned the following aspects of *Predicate Calculus*.

1. Variables (free and bound),
2. Predicates
3. And the quantifiers $\forall$ and $\exists$.

Think of the above as a check list: if you don't understand them then reread the notes, read a book, ask a questions, etc...

Good references: Chiswell & Hodges "Mathematical Logic" (Chapter 5).

Wiki for "Free variables and bound variables"

Tutorial

We now want to deduce some new equivalences. To help, we should recall some standard results from propositional logic:

$$(a \wedge b) \wedge c \equiv a \wedge (b \wedge c) \qquad (a \vee b) \vee c \equiv a \vee (b \vee c).$$

$$(a \wedge b) \vee c \equiv (a \vee c) \wedge (b \vee c) \qquad (a \vee b) \wedge c \equiv (a \wedge c) \vee (b \wedge c)$$

From these we can deduce:

1) $\forall x P(x) \wedge y \equiv \forall x \big(P(x) \wedge y\big)$.
2) $\forall x P(x) \vee y \equiv \forall x \big(P(x) \vee y\big)$.
3) $\exists x P(x) \wedge y \equiv \exists x \big(P(x) \wedge y\big)$.
4) $\exists x P(x) \vee y \equiv \exists x \big(P(x) \vee y\big)$.

Exercises

Next:

5) $\forall x\big(y \to P(x)\big) \ \equiv \ y \to \forall x P(x)$.
6) $\exists x(P(x) \to y) \ \equiv \ \forall x P(x) \to y$
7) $\exists x(P(x) \to y) \ \equiv \ \forall x P(x) \to y$
8) $\exists x\big(y \to P(x)\big) \ \equiv \ y \to \exists x P(x)$

. . . .

In the following cases, x is free in both P and Q. The equivalences are pretty easy to deduce.

9) $\forall x P(x) \land \forall x Q(x) \ \equiv \ \forall x\big(P(x) \land Q(x)\big)$
10) $\exists x P(x) \lor \exists x Q(x) \ \equiv \ \exists x\big(P(x) \lor Q(x)\big)$

(18/18) Further exercises:

Example: determine if the following "equivalences" are correct:

(i) $\forall x (P(x) \to Q(x)) \equiv \forall x P(x) \to \forall x Q(x)$.

(ii) $\forall x P(x) \vee \forall x Q(x) \equiv \forall x \big(P(x) \vee Q(x)\big)$

(iii) $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x \big(P(x) \wedge Q(x)\big)$