Mathematical and **Logic**al Aspects of Computing (CS304/CS310)

Lecture 18: Parse Trees (again); Equivalences

Tuesday, 5^{th} of November 2013

In Lecture 17 we considered the use of "parse trees" to visually represent expressions in propositional and predicate logic.
This will give us (another) way of expressing which instances of variables are "free" or "bound" in a given expression.
Equipped with that, we can go on to establish some equivalences involving statements in predicate logic.
(The approach for Parse Trees given here is based on Chiswell's and Hodges' textbook *Mathematical Logic*. The idea is essentially the same as the "inorder traversal of a tree" described briefly by Ben-Ari.).

(2/8) Parse trees in propositional logic

Last week we considered a few simple examples of parse trees for

$$a \wedge b; \qquad \neg a \vee b$$

. . . .

Now let's construct one for a more complicated, compound proposition:

$$\neg(a \to \neg b) \wedge (\neg a \to c)$$

 Parse trees for propositional statements

The procedure just exemplified can described as follows:

- Examine the expression to see if it contains
  - a unary operator or quantifier with an atomic proposition/symbol that can be replaced with a symbol. OR,
  - a binary operator applied to two atomic propositions/symbols that can be replaced with a symbol.
- If so, make these substitutions. Repeat until there is a single operator left.
- Produce the tree for this operator.
- Replace each symbol in turn with its original meaning. Expand these as trees.

(Note: this algorithm is to automate the procedure, and avoid error. You don't have to follow it to obtain the parse tree).

Examples:

1. $\forall x\big(P(x,y) \to \neg F(x)\big)$
2. $(x = y) \to \forall y\Bigg(\exists z\big(x = F(z,c,w)\big) \wedge P(y,z)\Bigg)$

(Note – this last example is taken from Chiswell and Hodges, *Mathematical Logic*, Example 7.2.2).

(4/8) Parse trees: free and bound variables

If a vertex label with a particular variable has an ancestor (parent, grandparent, etc) labeled with a quantifier applied to that variable, then that occurrence is **bound**. Otherwise it is **Free**.

Let's return again to the example:

$$(x = y) \rightarrow \forall y \Big(\exists z \big(x = F(z, c, w) \big) \wedge P(y, z) \Big)$$

(5/8) Quantifier equivalences (again)

Now that we have the fixed notion of free and bound variables, we can return to the topic of equivalences in predicate logic.
In Lecture 14 we saw how to use de Morgan's laws to deduce that

$$\neg\forall x P(x) \;\equiv\; \exists x \neg P(x). \qquad \neg\exists x P(x) \;\equiv\; \forall x \neg P(x).$$

Example: Give two ways of expressing "Not all who wander are lost" in the notation of predicate calculus.

(6/8) Quantifier equivalences (again)

In Lecture 14 we also saw how to use that

$$(a \wedge b) \wedge c \equiv a \wedge (b \wedge c) \qquad (a \vee b) \vee c \equiv a \vee (b \vee c).$$

$$(a \wedge b) \vee c \equiv (a \vee c) \wedge (b \vee c) \qquad (a \vee b) \wedge c \equiv (a \wedge c) \vee (b \wedge c)$$

to deduce:

1) $\forall x P(x) \wedge y \equiv \forall x \big(P(x) \wedge y\big)$.
2) $\forall x P(x) \vee y \equiv \forall x \big(P(x) \vee y\big)$.
3) $\exists x P(x) \wedge y \equiv \exists x \big(P(x) \wedge y\big)$.
4) $\exists x P(x) \vee y \equiv \exists x \big(P(x) \vee y\big)$.

Example:

(7/8) Quantifier equivalences (again)

Next:

5) $\forall x\big(y \to P(x)\big) \;\equiv\; y \to \forall x P(x)$. **Here is why:**

6) $\exists x(P(x) \to y) \;\equiv\; \forall x P(x) \to y$ **Here is why:**

7) $\exists x(P(x) \to y) \;\equiv\; \forall x P(x) \to y$

8) $\exists x\big(y \to P(x)\big) \;\equiv\; y \to \exists x P(x)$ (*Try 7+8 yourself*)

(8/8) If x is free in P and Q...

In the following cases, x is free in both P and Q. Here are some candidate for equivalences. Which are correct?

1) $\forall x P(x) \wedge \forall x Q(x) \equiv \forall x \big(P(x) \wedge Q(x)\big)$?

2) $\forall x P(x) \vee \forall x Q(x) \equiv \forall x \big(P(x) \vee Q(x)\big)$?

3) $\exists x P(x) \vee \exists x Q(x) \equiv \exists x \big(P(x) \vee Q(x)\big)$?

4) $\exists x P(x) \wedge \exists x Q(x) \equiv \exists x \big(P(x) \wedge Q(x)\big)$?