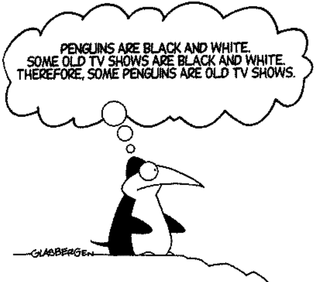
CS304/CS310: **LOGIC**

Lecture 19: More on Quantifier Equivalences



Logic: another thing that penguins aren't very good at.

Friday, the 8^{th} of November 2013

Last week we saw how to use de Morgan's laws to deduce that:

$$\neg\forall x P(x) \;\equiv\; \exists x \neg P(x). \qquad \neg\exists x P(x) \;\equiv\; \forall x \neg P(x).$$

We then went on to derive some new equivalences, such as those below. For all of them we suppose that x is *not* free in y. This means, for example, that $\forall x(y) = y$.

Then we deduced the following, among others:

- $\forall x P(x) \wedge y \;\equiv\; \forall x\big(P(x) \wedge y\big)$.
- $\exists x P(x) \vee y \;\equiv\; \exists x\big(P(x) \vee y\big)$.
- $\forall x\big(y \rightarrow P(x)\big) \;\equiv\; y \rightarrow \forall x P(x)$.
- $\exists x(P(x) \rightarrow y) \;\equiv\; \forall x P(x) \rightarrow y$

(3/1) If x is free in P and Q...

In the following cases, x is free in both P and Q. Here are some candidate for equivalences. Which are correct?

1) $\forall x P(x) \land \forall x Q(x) \;\equiv\; \forall x\big(P(x) \land Q(x)\big)$?

2) $\forall x P(x) \lor \forall x Q(x) \;\equiv\; \forall x\big(P(x) \lor Q(x)\big)$?

3) $\exists x P(x) \lor \exists x Q(x) \;\equiv\; \exists x\big(P(x) \lor Q(x)\big)$?

4) $\exists x P(x) \land \exists x Q(x) \;\equiv\; \exists x\big(P(x) \land Q(x)\big)$?

(4/1) Recall: Valid; Satisfiable

Recall some definitions from Propositional logic: if P is a proposition made from atomic propositions a and b, and the usual connectives, then:

- A particular choice of truth values to a and b is called an *assignment*. So there are four different assignments for the set $\{a, b\}$.
- An *interpretation* is some assignment applied to P. That is, picking values for a and b as they appear in P. An interpretation for which P is true is called a **model for P**.
- Saying *P is satisfiable* means there is some interpretation for which P is true.
- Saying *P is valid* (i.e., is a tautology) means that P is true for all interpretations.
- Saying *P is not satisfiable* (i.e., is a contradiction) means that P is false for all interpretations.
- Saying *P is not valid* (i.e., is a falsifiable) means that P is false for some interpretations.

(5/1) Recall: Valid; Satisfiable

Example 1: Show $\forall x P(x) \rightarrow P(a)$ is valid, where a is some fixed element in the universe of discourse.
Example 2: Is $\exists x P(x) \rightarrow P(a)$ is valid, where a is some fixed element in the universe of discourse?
Example 3: Is $P(a) \rightarrow \exists x P(x)$ is valid, where a is some fixed element in the universe of discourse?

(6/1) Semantic implication for predicates

Recall that in propositional calculus we would write $A \models B$ if every model for A is a model for B. We say the argument $A \models B$ is valid.
This idea extends to predicate calculus. The reasoning is more delicate, however. This, in part is because for propositional logic, we could write out all possible interpretations of A, establish which are models, and then check B for these. When working with predicates, this may not be feasible.
We'll study this in more detail next week. First, let's try to decide if the following arguments are valid:

(a) $\forall x\big(P(x) \to Q(x)\big) \models \forall x P(x) \to \forall x Q(x)$.

(b) $\forall x P(x) \to \forall x Q(x) \models \forall x\big(P(x) \to Q(x)\big)$.

(7/1) Semantic implication for predicates

We'll finish with the following famous example (mentioned in Lecture 14)

All men are mortal
Socrates is a man
Therefore, Socrates is mortal

Another popular one:

All roses are red
This flower is red
Therefore, it is a rose.