Mathematical and Logical Aspects of Computing (CS304/CS310)

Lecture 21: Semantic Tableaux for formulae with predicates

Friday, the 15^{th} of November 2013

Last week we reviewed some concepts in propositional logic:

(i) Consistency of a set of propositions.

(ii) Arguments of the form: $\{P_1, P_2, \ldots, P_n\} \models C$, and how this relates the to consistency of the set $\{P_1, P_2, \ldots, P_n, \neg C\}$

(iii) The method of *semantic tableaux* which can find a model for a set of propositions, or show no such model exists.

(iv) Based on (i)–(ii), it can also be used to establish the validity of an argument.

Reasoning with predicates and quantifiers is more subtle, not least because the domain of discourse may be very large, even infinite.

Keep the following in mind:

If we wish to show that the formula P is valid (i.e., always true), then try to find a model for $\neg P$. If this can't be done, then P is indeed valid.

(2/1) Semantic tableaux for predicate logic

We now wish to use semantic tableaux to establish if a formula (or set of formulae) in predicate calculus is satisfiable. The steps are

(i) Use de Morgan (for quantifiers) so negation only applies to literals or predicates, and not compounds or quantifiers. For example, consider the formula: $\neg\big(\forall x P(x) \to \forall x Q(x)\big)$. Rewrite this as:

(ii) *Instantiate* $\exists x P(x)$ as $P(a)$ where a is a term that has *not* be used so far. That is, since $\exists x P(x)$ means that $P(x)$ is true for *some* x in the universe of discourse, we'll call that a. If a has already been used, then use b, or c, or any term not used so far.

(iii) *Instantiate* $\forall x P(x)$ for *each* of the terms introduced so far.

(iv) Complete the tableau as usual; if a branch contains a literal and its negation, or a predicate with a constant and its negation, then close that branch.

(v) If all branches are closed, the formula is not satisfiable.

(3/1) Checking if P is valid by testing if $\neg P$ is satisfiable

Examples: We'll start with some formula for which we can easily determine (without a tableau) if they are satisfiable and if they are valid.

(i) $\{\forall x P(x) \wedge \exists x \neg P(x)\}$. (*Note: this is not even satisfiable*).

(ii) $\{\exists x P(x) \wedge \exists x \neg P(x)\}$. (*Note: this is satisfiable, but not valid*).

(iii) $\{\exists x P(x) \vee \exists x \neg P(x)\}$. (*Note: this is valid*).

(4/1) Checking if P is valid by testing if $\neg P$ is satisfiable

The ideas above can be applied to more complicated situations.
Example: Use the method of Semantic Tableaux to show that
$\forall x\big(P(x) \to Q(x)\big) \to \big(\forall x P(x) \to \forall x Q(x)\big)$ is valid.

(5/1) Checking if P is valid by testing if $\neg P$ is satisfiable

Now we'll use the same approach to show that the following formula is *not* valid, and to find a counter-example. $\forall x \big(P(x) \vee Q(x) \big) \rightarrow \big(\forall x P(x) \vee \forall x Q(x) \big)$

(6/1) Checking if P is valid by testing if $\neg P$ is satisfiable

Are the following sets consistent?

(1) $\{\forall x \big(P(x) \wedge Q(x)\big), \exists x \big(Q(x) \rightarrow \neg P(x)\big)\}$

(2) $\{\exists x P(x), \exists x \neg Q(x), \forall x \big(P(x) \wedge \neg Q(x)\big)\}$