Mathematical and Logical Aspects of Computing (CS304/CS310)

Lecture 22: Logical consequences, involving predicates and quantifiers

Tuesday, 19$^{\text{th}}$ of November 2013

Last week we studied the method of Semantic Tableaux for determining if a set of propositions/formulae is consistent or not, and the related idea of showing that a given proposition is valid.
Today, we'll extend this idea to determining if an argument is valid.
First, though, we'll revisit some more examples that we didn't get to during Friday's class.

 Recall: Semantic tableaux for predicate logic

The steps for forming a tableau are:

(i) Use de Morgan (for quantifiers) so negation only applies to literals or predicates, and not compounds or quantifiers.

(ii) *Instantiate* $\exists x P(x)$ as $P(a)$ where a is a term that has *not* be used so far. That is, since $\exists x P(x)$ means that $P(x)$ is true for *some* x in the universe of discourse, we'll call that a. If a has already been used, then use b, or c, or any term not used so far.

(iii) *Instantiate* $\forall x P(x)$ for *each* of the terms introduced so far.

(iv) Complete the tableau as usual; if a branch contains a literal and its negation, or a predicate with a constant and its negation, then close that branch.

(v) Any open branch gives a model, and so shows that the set is satisfiable. If all branches are closed, the formula is not satisfiable.

Example: use the tableau method to find out if the following sets consistent.

(1) $\{\forall x\big(P(x) \wedge Q(x)\big), \exists x\big(Q(x) \rightarrow \neg P(x)\big)\}$

(2) $\{\exists x P(x), \exists x \neg Q(x), \forall x\big(P(x) \wedge \neg Q(x)\big)\}$

A (single) proposition P is valid if it is true for all interpretations. We can write this as $\models P$. It is often useful to verify if P is (in)valid by checking if $\neg P$ is or is not satisfiable.

Example: use a semantic tableaux show that

$$\models \forall x \forall y P(x, y) \rightarrow P(a, a),$$

where a is any element of the universe of discourse.

Example: use the Tableaux approach to show that the following formula is *not* valid, and to find a counter-example. $\forall x\big(P(x) \vee Q(x)\big) \rightarrow \big(\forall x P(x) \vee \forall x Q(x)\big)$

An argument is valid if any model for the premises, $P_1, \ldots, P_n$ is also a model for the conclusion C. We write this as $\{P_1, \ldots, P_n\} \models C$

The argument $\{P_1, \ldots, P_n\} \models C$ is **valid** if $\{P_1, \ldots, P_n, \neg C\}$ is **inconsistent**.

Here are two classic examples – determine if the following arguments are valid:

(a) $\exists x \big(P(x) \rightarrow Q(x)\big) \models \exists x P(x) \rightarrow \exists x Q(x)$.

(b) $\exists x P(x) \rightarrow \exists x Q(x) \models \exists x \big(P(x) \rightarrow Q(x)\big)$.

Recall: Universe of discourse

We think of a predicate as a function that maps elements from some set to the truth values $\{F, T\}$. The domain of this predicate is often called the *universe of discourse*.

. . . .

Translating natural language to 1st order logic

There is not just one right way to do this, but also many wrong ways. To be correct we must:

1. state clearly the universe of discourse;
2. define any functions we need (or better...)
3. define any predicates we need
4. apply quantifiers appropriately.

(8/8) Reasoning with predicates and quantifiers

Consider the following argument, taken from CS304 Summer Exam 2011/2012:

> *All roses smell nice. This flower is a rose. Therefore this flower smells nice.*

. . . .

Is the following argument valid?

> *All penguins are black and white. All old TV shows are black and white. All that I can think of, is either a penguin or an old TV show. Therefore, some penguin that I can think of, is an old TV show.*