CS304/CS310 – Mathematical and Logical Aspects of Computing

Problem Set 4

1. The notation $\exists! x P(x)$ denotes the proposition

   There exists a unique x such that $P(x)$ is true.[1]

   Express $\exists! x P(x)$, where the universe of discourse consists of the integers 1, 2, and 3, in terms of negation, conjunctions and disjunctions.

2. If the universe of discourse consists of the integers, $\mathbb{Z}$, what are the truth values of the following propositions?

   (i) $\exists! x(x > 1)$.
   (ii) $\exists! x(x^2 = 1)$.
   (iii) $\exists! x(x + 3 = 2x)$.
   (iv) $\exists! x(x = x + 1)$.
   (v) $\exists! x P(x) \to \exists x P(x)$

3. Use de Morgan's Laws to show that $\neg\exists x P(x) \equiv \forall x \neg P(x)$. (For simplicity, you may assume that the universe of discourse has only three elements).

4. Rewrite each of the following statements so that negations appear only within predicates (that is, negation should not be applied immediately to the left of a quantifier symbol, or to a compound proposition.

   a) $\neg\exists y \exists x P(x, y)$.
   Solution: $\neg\exists y\Big(\exists x P(x)\Big) \equiv \forall y\Big(\neg\exists x P(x)\Big) \equiv \forall y \forall x \neg P(x)$, *as required*
   b) $\neg\forall x \exists y P(x, y)$.
   c) $\neg\exists y\Big(Q(y) \wedge \forall x \neg R(x, y)\Big)$.
   d) $\neg\exists y\Big(\exists x R(x, y) \vee \forall x S(x, y)\Big)$.

5. Use a semantic tableau to show that the following argument is valid:
   $\forall x(P(x) \to Q(x))$ yields $\forall x P(x) \to \forall x Q(x)$.
   Does the converse hold, i.e., is it true that
   $\forall x P(x) \to \forall x Q(x)$ yields $\forall x(P(x) \to Q(x))$?
   Explain your answer.

[1] This question is based on an exercise in Section 1.3 of Rosen "Discrete Mathematics and Its Applications", 5th Ed. So too are Questions 2 and 4