A Framework for Computing Zeta Functions of Groups, Algebras, and Modules

Tobias Rossmann

Abstract We give an overview of the author's recent work on methods for explicitly computing various types of zeta functions associated with algebraic counting problems. Among the types of zeta functions that we consider are the so-called topological ones.

Key words: Subgroup growth, representation growth, zeta functions, topological zeta functions, unipotent groups, p-adic integration, Newton polytopes
Subject Classifications:11M41, 20F69, 14M25, 20F18, 20C15, 20G30

1 Introduction

1.1 Zeta functions in group theory and related fields

The past decades saw the development of a theory of zeta functions of groups and related algebraic structures. In this article, we consider subobject and representation zeta functions related to enumerative problems associated with nilpotent groups. For introductions to the area and surveys of developments in particular directions, we refer the reader to [15, 19, 29, 56, 57]. We will concern ourselves with zeta functions that one can attach to a suitable infinite algebraic object (e.g. a Lie algebra or a group). In a different direction, zeta functions have found striking applications in the study of infinite families of finite groups; for surveys of this active branch of asymptotic group theory, we refer to [34, 46].

Tobias Rossmann
Department of Mathematics, University of Auckland, Auckland, New Zealand
e-mail: tobias.rossmann@gmail.com

The subobject zeta functions of interest to us can be traced back to a number of sources. An early ancestor is given by the Dedekind zeta function of a number field, an instance of a submodule zeta function as defined below. More recently, L. Solomon [48] introduced zeta functions enumerating $\mathbf{Z}G$-lattices within a fixed $\mathbf{Z}G$-module for a finite group G. In a seemingly different direction, a hugely influential paper of Grunewald, Segal, and Smith [23] initiated the study of zeta functions arising from the enumeration of subgroups of finite index in a given finitely generated torsion-free nilpotent group (a $\mathcal{T}$-*group*, for short). In detail, given such a group G, they defined its *(global) subgroup zeta function* to be

$$\zeta_G^{\leqslant}(s) = \sum_H |G:H|^{-s},$$

where H ranges over the subgroups of finite index of G. They also established various key properties of these zeta functions such as:

- (Convergence.) Let h be the Hirsch length of G. Then $\zeta_G^{\leqslant}(s)$ converges and defines an analytic function on the half-plane $\{s \in \mathbf{C} : \mathrm{Re}(s) > h\}$.
- (Euler product.) $\zeta_G^{\leqslant}(s) = \prod\limits_{p \text{ prime}} \zeta_{\hat{G}_p}^{\leqslant}(s)$, where $\hat{G}_p$ denotes the pro-p completion of G and each *local subgroup zeta function* $\zeta_{\hat{G}_p}^{\leqslant}(s)$ is defined by enumerating open subgroups of $\hat{G}_p$ according to their indices.
- (Rationality.) Each $\zeta_{\hat{G}_p}^{\leqslant}(s)$ is rational in p^{-s} over $\mathbf{Q}$.

For $G = \mathbf{Z}$, we recover the Riemann zeta function $\zeta(s) = \zeta_{\mathbf{Z}}(s) = \sum_{n=1}^{\infty} n^{-s}$ and the classical Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$. This simple illustration notwithstanding, while the first two of the above points are elementary, the rationality of local subgroup zeta functions is a deep theorem.

By only considering normal subgroups of finite index of G, the *normal subgroup zeta function* $\zeta_G^{\triangleleft}(s)$ of G is obtained; it satisfies the evident analogues of the properties stated above. The subalgebra and submodule zeta functions defined in §2.1 essentially constitute generalisations of the local and global (normal) subgroup zeta functions associated with nilpotent groups. Indeed, as explained in [23], the Mal'cev correspondence allows us to linearise the enumeration of subgroups by replacing the nilpotent group in question by a suitable nilpotent Lie $\mathbf{Z}$-algebra (at the cost of having to discard finitely many Euler factors).

Apart from subobject zeta functions, we also consider representation zeta functions. These are Dirichlet series enumerating certain finite-dimensional irreducible representations of a suitable group up to adequate notions of equivalence. Representation zeta functions were introduced by Witten [59] in the context of complex Lie groups. Jaikin-Zapirain [26] made fundamental contributions to the study of representation zeta functions of compact p-adic analytic groups. Within infinite group theory, a substantial amount of recent work has been devoted to representation zeta functions of groups arising from semisimple algebraic groups; see e.g. work of Larsen and Lubotzky [32] and

Avni et al. [2]. In a seemingly different direction, Hrushovski and Martin [24] (v1, 2006) introduced representation zeta functions of $\mathcal{T}$-groups; these are the representation zeta functions that we shall consider.

As we will explain in §3, the subobject and representation zeta functions considered here share a crucial common feature: in each case, a single global object (e.g. a $\mathcal{T}$-group) gives rise to a family of associated local zeta functions indexed by primes (or places of a number field) and, after excluding finitely many exceptions, these local zeta functions can all be described in terms of a single "formula". We express this by saying that there exists such a "formula" for the *generic* local zeta functions in question. In a surprising number of interesting cases (including most cases that have been successfully computed so far), the local zeta functions under considerations are in fact given by a rational function $W(p, p^{-s})$ in p and p^{-s} over $\mathbf{Q}$ (again after possibly excluding finitely many primes). This phenomenon is referred to as *(almost) uniformity* in [19, §1.2.4]. In such uniform cases, we interpret the natural task of computing the generic local zeta functions under consideration as computing the (uniquely determined) rational function W. For instance, if $H(\mathbf{Z})$ is the discrete Heisenberg group, then, by [23, Prop. 8.1],

$$\zeta^{\leqslant}_{H(\mathbf{Z})}(s) = \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)\zeta(3s-3)^{-1} = \prod_{p \text{ prime}} W(p, p^{-s}), \quad (1)$$

where $W(X,Y) = (1-X^3Y^3)/\big((1-Y)(1-XY)(1-X^2Y^2)(1-X^3Y^2)\big)$.

1.2 Computations: limitations and previous work

Theoretical results on the subobject and representation zeta functions considered here frequently rely on impractical or even non-constructive methods. In particular, in one of the central papers in the area, du Sautoy and Grunewald [16] showed that generic local subobject zeta functions are in principle "computable" (in the sense that one can compute certain formulae for them, see §3)—provided that one happens to know an embedded resolution of singularities of some (usually highly singular) hypersurface inside some affine space (of dimension $\geqslant 6$ in all cases of interest); Voll [55, §3.4] obtained a similar result for representation zeta functions of nilpotent groups.

Apart from striking theoretical applications, the methods developed by du Sautoy and Grunewald [16] and Voll [55] (and Stasinski and Voll [49]) have also been successfully used to compute zeta functions in small examples (see, in particular, the computation of du Sautoy and Taylor [18] of the subalgebra zeta function of $\mathfrak{sl}_2(\mathbf{Z})$; for related computations, see [14, 25, 30]). However, when it comes to explicit computations, the practical scope of these techniques is usually rather limited.

A substantial number of subobject zeta functions (primarily of nilpotent groups and Lie algebras) were computed by Woodward [60]. He relied on a combination of human guidance and computer calculations. Unfortunately, due to a lack of documentation, his findings are hard to reproduce. A number of ad hoc computations of representation zeta functions of nilpotent groups have been carried out by Ezzat [22], Snocken [47], and Stasinski and Voll [50].

1.3 Topological zeta functions

A common catchphrase in the area is that topological zeta functions are obtained from local ones (such as the $\zeta^{\leqslant}_{\hat{G}_p}(s)$ from above) by passing to a limit "$p \to 1$". Indeed, Denef and Loeser [13] introduced topological zeta functions of polynomials by justifying that such a limit can be applied to Igusa's local zeta function (see [11, 36] for introductions). Despite their arithmetic ancestry (for Igusa's local zeta function enumerates solutions to congruences), research on topological zeta functions has been primarily motivated by questions from singularity theory. In recent years, topological zeta functions of polynomials have mostly been studied within the realm of motivic zeta functions.

Using such a "motivic" point of view, topological subobject zeta functions were introduced by du Sautoy and Loeser [17]; these zeta functions are related to, but different from, Evseev's "reduced zeta functions" [21]. Apart from giving a definition of these zeta functions, they also computed a few small examples. Further examples were determined by the author [38, 39] who also began an investigation of topological representation zeta functions [40]. The topological subobject and representation zeta functions studied by the author seem to exhibit a number of features distinct from the well-studied case of topological zeta functions of polynomials; we will discuss some of these features in §8.

1.4 Computations: a framework

The author's articles [38–41] provide a practical framework for explicitly computing numerous types of (generic) local and topological zeta functions in "fortunate" cases related to geometric genericity conditions. The main purpose of the present article is to provide a self-contained and unified introduction that takes into account theoretical developments that occurred over the course of the project.

In summary, the author's methods for computing topological [38–40] or generic local zeta functions [41] all proceed along the following lines:

1. (Translation.) Express the associated generic local zeta functions in terms of p-adic integrals defined in terms of certain global "data".

2. (The simplify-balance-reduce loop.) After discarding finitely many primes, attempt to write the integrals from the first step as sums of integrals of the same shape but defined in terms of "regular" (i.e. sufficiently generic) data.
3. (Evaluation.) Assuming the second step succeeds, explicitly compute "formulae" for the generic local or topological zeta functions associated with the integrals attached to the regular data from the second step.
4. (Final summation.) Add the formulae from the third step.

The first step is based on known results. For the computation of subobject zeta functions, we use the formalism of "cone integrals" of du Sautoy and Grunewald [16]. For representation zeta functions associated with unipotent groups, we rely on the formulation in terms of p-adic integrals due to Stasinski and Voll [49] (which extends Voll's formalism from [55]); see also related work of Avni et al. [2] on representation zeta functions of arithmetic groups.

The second step is concerned with manipulations of p-adic integrals represented in terms of the "toric data" from [39] or the "representation data" from [40]. Either type of datum consists of algebraic ingredients (Laurent polynomials) and convex-geometric data ("half-open cones"). Regularity is an algebro-geometric genericity condition which allows us to invoke the machinery developed in [38] in order to compute the p-adic integral in question (or the associated topological zeta function). Being "balanced" is a much weaker property and it is always possible to write the integral associated with an arbitrary toric/representation datum as a sum of integrals associated with balanced data—this corresponds to the middle part of the name of the second step. In fortunate case (related to the notion of non-degeneracy from [38]), applying the balancing procedure to our initial datum from the first step will produce a family of regular data. The purpose of the reduction step is to modify balanced but singular (i.e. not regular) data, the goal being to derive regular data. This may or may not succeed for a given example and it is the main reason why the author's methods may fail for specific examples. While the final summation step is mathematically trivial, it is often computationally daunting.

1.5 Overview

In §2, we recall definitions of the global and local zeta functions that we consider. We then discuss the existence of "formulae" for generic local zeta functions in §3. Topological zeta functions are the subject of §4. Prior to presenting our computational framework and its computer implementation in §6, we collect some background material from convex geometry in §5. As a demonstration of the practical usefulness of the author's methods, a number of applications are discussed in §7. Finally, §8 is devoted to two particularly interesting conjectures that arose from the author's computations.

2 Global and local zeta functions

2.1 Formal subalgebra, ideal, and submodule zeta functions

Let R be a commutative ring and let A be an R-algebra, i.e. an R-module endowed with a multiplication $\mathsf{A} \otimes_R \mathsf{A} \to \mathsf{A}$ (which need not be associative or Lie). A *subalgebra* of A is an R-submodule which is stable under the given multiplication. As usual, by the *index* $|\mathsf{A} : \mathsf{U}|$ of an R-submodule $\mathsf{U} \leqslant \mathsf{A}$, we mean the cardinality of the R-module quotient A/U. Let $a_n^{\leqslant}(\mathsf{A})$ denote the number of subalgebras of A of index n. Assuming that these numbers are all finite, we define the *subalgebra zeta function* of A to be the formal Dirichlet series

$$\zeta_{\mathsf{A}}^{\leqslant}(s) = \sum_{n=1}^{\infty} a_n^{\leqslant}(\mathsf{A}) n^{-s}.$$

If we only consider (2-sided R-)ideals of A, then we obtain the *ideal zeta function* $\zeta_{\mathsf{A}}^{\triangleleft}(s)$ of A. These notions are all natural generalisations of the subring and ideal zeta functions introduced by Grunewald, Segal, and Smith [23].

Let M be an R-module and let Ω be a set of endomorphisms of M. Let $a_n(\Omega \curvearrowright \mathsf{M})$ denote the number of submodules U of M with $|\mathsf{M} : \mathsf{U}| = n$ and such that U is invariant under each element of Ω. Assuming that each $a_n(\Omega \curvearrowright \mathsf{M})$ is finite, we define the *submodule zeta function* of Ω acting on M to be

$$\zeta_{\Omega \curvearrowright \mathsf{M}}^{\leqslant}(s) = \sum_{n=1}^{\infty} a_n(\Omega \curvearrowright \mathsf{M}) n^{-s}.$$

These zeta functions generalise those of Solomon [48]. It is frequently useful to note that $\zeta_{\Omega \curvearrowright \mathsf{M}}(s)$ only depends on the unitary associative subalgebra of $\mathrm{End}(\mathsf{M})$ generated by Ω. Moreover, as pointed out in [38], submodule zeta functions as defined here generalise the ideal zeta functions from above.

Generalising further, we could take into account a given R-module decomposition of an R-algebra A or an R-module M and consider associated graded counting problems as in [41, §3]; apart from the author's work, such graded zeta functions have recently been studied by Lee and Voll [33]. For the sake of simplicity, while many results and ideas apply in this greater generality, in the following, we only consider subalgebra and submodule zeta functions of the form $\zeta_{\mathsf{A}}^{\leqslant}(s)$ and $\zeta_{\Omega \curvearrowright \mathsf{M}}(s)$ and we refer to these as *subobject zeta functions*.

2.2 Number fields and Euler products

The subalgebra and submodule zeta functions defined in §2.1 are formal Dirichlet series. Further assumptions are needed for these to give rise to analytic functions.

We first set up some notation that will be used for the remainder of this article. Let k be a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{V}_k$ be the set of non-Archimedean places of k; we identify $\mathcal{V}_{\mathbf{Q}}$ with the set of prime numbers. For $v \in \mathcal{V}_k$, let $\mathfrak{p}_v \in \mathrm{Spec}(\mathfrak{o})$ correspond to v, let k_v denote the v-adic completion of k, and let $\mathfrak{o}_v$ be the valuation ring of k_v. We write $\mathfrak{K}_v$ for the residue field of k_v and q_v for its size.

For an $\mathfrak{o}$-object (e.g. an $\mathfrak{o}$-module) X, we write X_v for the associated $\mathfrak{o}_v$-object obtained after base change (e.g. $\mathsf{X} \otimes_{\mathfrak{o}} \mathfrak{o}_v$). Let A be an $\mathfrak{o}$-algebra, let M be an $\mathfrak{o}$-module, and let $\Omega \subseteq \mathrm{End}(\mathsf{M})$. We assume that A and M are both free of rank d as $\mathfrak{o}$-modules. It is well-known (cf. [23, Prop. 1]) that $\zeta_{\mathsf{A}}^{\leqslant}(s)$ and $\zeta_{\Omega \curvearrowright \mathsf{M}}(s)$ both converge for $\mathrm{Re}(s) > d$. Furthermore, we obtain Euler products

$$\zeta_{\mathsf{A}}^{\leqslant}(s) = \prod_{v \in \mathcal{V}_k} \zeta_{\mathsf{A}_v}^{\leqslant}(s), \qquad \zeta_{\Omega \curvearrowright \mathsf{M}}(s) = \prod_{v \in \mathcal{V}_k} \zeta_{\Omega \curvearrowright \mathsf{M}_v}(s);$$

see [38, Lem. 2.3]. In [16], du Sautoy and Grunewald showed that $\zeta_{\mathsf{A}}^{\leqslant}(s)$ and $\zeta_{\Omega \curvearrowright \mathsf{M}}(s)$ have rational abscissae of convergence and admit meromorphic continuation to some larger half-planes than their initial half-planes of convergence. Furthermore, using their techniques (or the model-theoretic arguments from [23]), each $\zeta_{\mathsf{A}_v}^{\leqslant}(s)$ and $\zeta_{\Omega \curvearrowright \mathsf{M}_v}(s)$ is found to be rational in q_v^{-s}.

2.3 Representation zeta functions of unipotent groups

Given a topological group G, let $\tilde{r}_n(G)$ denote the number of its continuous irreducible n-dimensional complex representations, counted up to equivalence and tensoring with continuous 1-dimensional representations ("twisting"). The motivation for allowing 1-dimensional "twists" comes from the case of nilpotent groups: while an infinite (discrete) $\mathcal{T}$-group G has infinitely many homomorphisms to $\mathrm{GL}_1(\mathbf{C})$, Lubotzky and Magid [35] showed that each $\tilde{r}_n(G)$ is finite. Following Hrushovski and Martin [24] (v1), if each $\tilde{r}_n(G)$ is finite, we define the *(twist) representation zeta function* of G to be the formal Dirichlet series

$$\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}.$$

Let G be a $\mathcal{T}$-group. Then $\zeta_G^{\widetilde{\mathrm{irr}}}(s)$ converges in some complex half-plane, see [49, Lem. 2.1]. Moreover, crucial properties of $\zeta_G^{\widetilde{\mathrm{irr}}}(s)$ such as its abscissa of convergence only depend on the commensurability class of G, see [20, Cor. B]. It is well-known that commensurability classes of $\mathcal{T}$-groups are in natural bijection with isomorphism classes of unipotent algebraic groups over $\mathbf{Q}$. Following Stasinski and Voll [49], we consider representation zeta functions of $\mathcal{T}$-groups associated with unipotent algebraic groups over number fields.

We first recall some facts on unipotent algebraic groups. Let U_d be the subgroup scheme of GL_d consisting of upper unitriangular matrices. An algebraic group $\mathbf{G}$ over the number field k is *unipotent* if and only if it embeds into some $\mathrm{U}_d \otimes k$; for other characterisations of unipotence, see [10, Ch. IV]. Let $\mathbf{G}$ be a unipotent algebraic group over k. After choosing an embedding of $\mathbf{G}$ into some $\mathrm{U}_d \otimes k$, we obtain an associated $\mathfrak{o}$-*form* G of $\mathbf{G}$ as a group scheme by taking the scheme-theoretic closure of $\mathbf{G}$ within $\mathrm{U}_d \otimes \mathfrak{o}$. We regard the $\mathcal{T}$-group $\mathsf{G}(\mathfrak{o})$ as a discrete topological group and for $v \in \mathcal{V}_k$, we naturally regard $\mathsf{G}(\mathfrak{o}_v)$ as a pro-p_v group, where p_v is the rational prime contained in $\mathfrak{p}_v$. By [49, Prop. 2.2], $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o})}(s) = \prod_{v \in \mathcal{V}_k} \zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o}_v)}(s)$. Duong and Voll [20] and Hrushovski et al. [24] have shown that $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o}_v)}(s)$ is rational in q_v^{-s} for almost all $v \in \mathcal{V}_k$ and that $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o})}(s)$ has rational abscissa of convergence. Duong and Voll also showed that, as in the enumeration of subobjects in §2.2, $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o})}(s)$ admits meromorphic continuation to the left of its abscissa of convergence.

3 Computability of generic local zeta functions

We now explain in which sense generic local subobject and representation zeta functions are, in principle, computable. Let $\mathsf{Z} = (\mathsf{Z}_v(s))_{v \in \mathcal{V}_k}$ be a family of local zeta functions defined in one of the following ways:

- $\mathsf{Z}_v(s) = (1 - q_v^{-1})^d \cdot \zeta^{\leqslant}_{\mathsf{A}_v}(s)$, where A is an $\mathfrak{o}$-algebra whose underlying $\mathfrak{o}$-module is free of rank d.
- $\mathsf{Z}_v(s) = (1 - q_v^{-1})^d \cdot \zeta_{\Omega \curvearrowright \mathsf{M}_v}(s)$, where M is a free $\mathfrak{o}$-module of rank d and $\Omega \subseteq \mathrm{End}(\mathsf{M})$.
- $\mathsf{Z}_v(s) = \zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G}(\mathfrak{o}_v)}(s)$, where $\mathsf{G} \leqslant \mathrm{U}_d \otimes \mathfrak{o}$ is the natural $\mathfrak{o}$-form of $\mathbf{G} \leqslant \mathrm{U}_d \otimes k$.

The role of the factors $(1 - q_v^{-1})^d$ will be explained in §4. The global zeta function associated with Z is in general a subtle analytic object which we shall not consider further. Instead, we focus on the already quite difficult local picture.

In the study of local zeta functions $\mathsf{Z}_v(s)$ attached to a global object, the exclusion of finite sets of exceptional places is often unavoidable. For example, while the subalgebra zeta function of $\mathfrak{sl}_2(\mathbf{Z}_p)$ is given by a simple formula which is valid for all odd primes p, the case $p = 2$ is exceptional; see [18]. Fortunately, interesting properties of global zeta functions often remain unaffected when finitely many places are dropped; this is, for instance, the case for the global abscissae of convergence of subobject zeta functions, cf. [42, Lem. 5.3, Rem. 5.4]. Henceforth, we focus on the *generic* local zeta functions $\mathsf{Z}_v(s)$ obtained after discarding $\mathsf{Z}_w(s)$ for finitely many $w \in \mathcal{V}_k$.

As we mentioned above, $\mathsf{Z}_v(s)$ is rational in q_v^{-s} for almost every $v \in \mathcal{V}_k$. The task of "computing" $\mathsf{Z}_v(s)$ then means to determine $W_v(Y) \in \mathbf{Q}(Y)$ with $\mathsf{Z}_v(s) =$

$W_v(q_v^{-s})$. The non-trivial fact that it is even possible to do this algorithmically is a consequence of the proof of the following deep theorem.

Theorem 1. *Let $\mathsf{Z} = (\mathsf{Z}_v(s))_{v\in\mathcal{V}_k}$ be a family of local subalgebra, submodule, or representation zeta functions as above. There are k-varieties $V_1,\dots,V_r$ and rational functions $W_1(X,Y),\dots,W_r(X,Y) \in \mathbf{Q}(X,Y)$ such that for almost all $v \in \mathcal{V}_k$,*

$$\mathsf{Z}_v(s) = \sum_{i=1}^{r} \#\bar{V}_i(\mathfrak{K}_v)\cdot W_i(q_v, q_v^{-s}), \tag{2}$$

where $\bar{V}_i$ denotes the reduction modulo $\mathfrak{p}_v$ of a fixed $\mathfrak{o}$-model of V_i.

Proof. For subalgebra and subobject zeta functions, this is due to du Sautoy and Grunewald [16] (cf. [38, Ex. 5.11(iii)]). For representation zeta functions associated with unipotent groups, it was proved by Stasinski and Voll [49, Pf of Thm A] (building upon previous work of Voll [55, §3.4]).

Remark 1. A seemingly stronger version of Theorem 1 is given by [41, Thm 4.1]. This strengthened version takes into account not only the variation of the place v but also allows local base extensions in a suitable manner. However, by [43], the validity of (2) under variation of v (excluding finitely many exceptions) already implies the validity of its analogues after local base extensions. This observation allows us to rephrase some of our previous results more concisely in the present article.

While the proofs of Theorem 1 in the sources cited above are constructive, they all rely on some form of resolution of singularities for k-varieties; for non-constructive model-theoretic approaches, see e.g. [24, 37]. Even though algorithms for constructive resolution of singularities are known (see [7, 54]), these are typically impractical in the present context. Nonetheless, we obtain an algorithm which computes $\mathsf{Z}_v(s)$, for each $v \in \mathcal{V}_k$ outside of some finite set, as a rational function in q_v^{-s}. It is tempting to regard the explicit construction of (2) as the simultaneous computation of all generic local zeta functions $\mathsf{Z}_v(s)$ at once. This point of view, however, is not entirely satisfactory. For instance, it is unclear how to decide if two formulae of the form (2) define the same rational function for almost all $v \in \mathcal{V}_k$.

For many examples of interest, the phenomenon of "uniformity" mentioned in the introduction allows us to bypass such problems. We say that Z is *uniform* if there exists $W(X,Y) \in \mathbf{Q}(X,Y)$ such that $\mathsf{Z}_v(s) = W(q_v, q_v^{-s})$ for almost all $v \in \mathcal{V}_k$. While the author is not aware of any method for testing uniformity of Z, if it is indeed uniform, our goal is to compute $W(X,Y)$.

Before we proceed further with our work towards this goal, the author would like to emphasise two points. First, he is not aware of a better general notion of computing generic local zeta functions than to construct a formula (2) (or a motivic analogue as in [17]). Secondly, he is not aware of a method for carrying out such a construction which is both general and practical. These two points explain why the author's practical methods for computing generic local zeta functions, described in §6 below, are not general, i.e. they will not succeed in all cases.

4 Topological zeta functions

We now introduce the protagonist of [38–40]: topological zeta functions. These functions are defined analogously to topological zeta functions of polynomials, as introduced by Denef and Loeser [13]; topological subobject zeta functions were first defined by du Sautoy and Loeser [17].

4.1 An informal "definition"

Informally, we obtain the topological zeta function associated with a family $\mathsf{Z} = (\mathsf{Z}_v(s))_{v \in \mathcal{V}_k}$ as in Theorem 1 by taking the limit "$q_v \to 1$", obtained as the constant term in the binomial expansion of a "generic" $\mathsf{Z}_v(s)$ as a series in $q_v - 1$. For example, by [49, Thm B], if $\mathsf{H} = \mathrm{U}_3$ is the Heisenberg group scheme, then for each $v \in \mathcal{V}_k$,

$$\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{H}(\mathfrak{o}_v)}(s) = \frac{1 - q_v^{-s}}{1 - q_v^{1-s}} \tag{3}$$

and by symbolically expanding

$$q_v^{a-bs} = (1 + (q_v - 1))^{a-bs} = \sum_{\ell=0}^{\infty} \binom{a-bs}{\ell} (q_v - 1)^{\ell},$$

we obtain $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{H}(\mathfrak{o}_v)}(s) = \frac{s}{s-1} + \mathscr{O}(q_v - 1)$ whence the topological representation zeta function of H is $\zeta_{\mathsf{H},\mathrm{top}}(s) = s/(s-1)$. By [43, §4], this informal "definition" of topological zeta functions is rigorous in uniform cases such as (3). However, the author is not aware of a definition of topological zeta functions which is at the same time elementary, general, rigorous, and short. A pragmatic motivation for studying topological zeta functions is that they turn out to be the type of mathematical invariant which, while hard to define, can often be computed and studied effectively. Moreover, as observed by Denef and Loeser [13, Thm 2.2], by the very nature of the limit "$q_v \to 1$" used to define them, topological zeta functions preserve interesting analytic properties of their local relatives.

4.2 A rigorous definition

Nowadays, topological zeta functions are most commonly studied in the context of motivic zeta functions and integrals. In contrast, the following exposition is based on the author's axiomatisation [38, §5] of the original "arithmetic" definition of the topological zeta function of a polynomial by Denef and Loeser [13].

A rigorous notion of the limit "$q_v \to 1$" is based on a formula (2). Specifically, we define such a limit separately for the terms "$\#\bar{V}_i(\mathfrak{K}_v)$" and "$W_i(q_v, q_v^{-s})$" and then combine them in the evident way.

First, we formalise taking a limit "$q_v \to 1$" of $W(q_v, q_v^{-s})$. For $e \in \mathbf{Q}[s]$, write $X^e := \sum_{\ell=0}^{\infty} \binom{e}{\ell} (X-1)^{\ell} \in \mathbf{Q}[s][\![X-1]\!]$. The map $f(X,Y) \mapsto f(X, X^{-s})$ yields an embedding of $\mathbf{Q}(X,Y)$ into $\mathbf{Q}(s)((X-1))$. In general, $W(X, X^{-s})$ need not be a power series in $X-1$ for $W(X,Y) \in \mathbf{Q}(X,Y)$. We will restrict attention to certain rational functions for which it is:

Definition 1.

1. Let $\mathbf{M}[X,Y] \subseteq \mathbf{Q}(X,Y)$ be the $\mathbf{Q}$-algebra consisting of those rational functions $W(X,Y) \in \mathbf{Q}(X,Y)$ with $W(X,X^{-s}) \in \mathbf{Q}(s)[\![X-1]\!]$ and such that $W(X,Y) = f(X,Y)/\big((1-X^{a_1}Y^{b_1}) \cdots (1-X^{a_r}Y^{b_r})\big)$ for non-zero $(a_1,b_1), \dots, (a_r,b_r) \in \mathbf{Z}^2$ and a suitable $f(X,Y) \in \mathbf{Q}[X^{\pm 1}, Y^{\pm 1}]$.
2. Write $\lfloor W(s) \rfloor$ for the image of $W(X,Y) \in \mathbf{M}[X,Y]$ under "formal reduction modulo $X-1$", i.e. under the map $f(X,Y) \mapsto f(X,X^{-s}) \bmod (X-1)$.

The factors $(1-q_v^{-1})^d$ in the definition of $\mathsf{Z}_v(s)$ in §3 were included to ensure the validity of the following:

Lemma 1. *We may assume that $W_1(X,Y), \dots, W_r(X,Y) \in \mathbf{M}[X,Y]$ in Theorem 1.*

Proof. Combine [38, Thm 5.16] and [40, Lem. 3.4].

It remains to define a limit "$q_v \to 1$" of $\#\bar{V}_i(\mathfrak{K}_v)$ in (2). For background and further details on the following, we refer to [45, §4]. For $v \in \mathcal{V}_k$, fix an algebraic closure $\bar{\mathfrak{K}}_v$ of $\mathfrak{K}_v$ and denote by $\mathfrak{K}_v^{(f)}$ the extension of $\mathfrak{K}_v$ of degree f within $\bar{\mathfrak{K}}_v$. Let V be a k-variety. As above, we fix an $\mathfrak{o}$-model, V say, of V and given $v \in \mathcal{V}_k$, we let $\bar{V}$ denote the reduction modulo $\mathfrak{p}_v$ of V. It follows from Grothendieck's trace formula and comparison theorems for ℓ-adic cohomology that for almost all $v \in \mathcal{V}_k$, there are finitely many non-zero complex numbers α_{ij} $(i,j \geqslant 0)$ such that for all $f \in \mathbf{N}$, $\#\bar{V}(\mathfrak{K}_v^{(f)}) = \sum_{i,j} (-1)^i \alpha_{ij}^f$ and, moreover, $\#\bar{V}(\mathfrak{K}_v^{(0)}) := \sum_{i,j}(-1)^i \alpha_{ij}^0 = \chi(V(\mathbf{C}))$; here, the topological Euler characteristic $\chi(V(\mathbf{C}))$ is taken with respect to an arbitrary embedding of k into $\mathbf{C}$. Numerous results in [45] justify defining $\#\bar{V}(\mathfrak{K}_v^{(0)})$ as $\chi(V(\mathbf{C}))$. For example, by [41, Lem. 7.1] (an application of Chebotarev's density theorem similar to arguments from [45]), if $f(X) \in \mathbf{Z}[X]$ satisfies $\#\bar{V}(\mathfrak{K}_v) = f(q_v)$ for almost all $v \in \mathcal{V}_k$, then $\chi(V(\mathbf{C})) = f(1)$.

In summary, our candidate for the topological zeta function associated with a family $\mathsf{Z} = (\mathsf{Z}_v(s))_{v\in\mathcal{V}_k}$ as in Theorem 1 is $\sum_{i=1}^r \chi(V_i(\mathbf{C}))\cdot\lfloor W_i(s)\rfloor \in \mathbf{Q}(s)$. It remains to show that this rational function does not depend on the choice of the particular formula (2). This is the content of the following theorem.

Theorem 2. *For $v \in \mathcal{V}_k$, let $\mathsf{Z}_v(s)$ be an analytic function on some complex right half-plane. Let $V_1,\dots,V_r$ be k-varieties, let $W_1(X,Y),\dots,W_r(X,Y) \in \mathbf{M}[X,Y]$, and suppose that for almost all $v \in \mathcal{V}_k$, $\mathsf{Z}_v(s) = \sum\limits_{i=1}^{r} \#\bar{V}_i(\mathfrak{K}_v)\cdot W_i(q_v,q_v^{-s})$. Then the following rational function is independent of the V_i and the $W_i(X,Y)$:*

$$\mathsf{Z}_{\mathrm{top}}(s) := \sum_{i=1}^{r} \chi(V_i(\mathbf{C}))\cdot\lfloor W_i(s)\rfloor \in \mathbf{Q}(s).$$

Proof. Combine [38, Thm 5.12] and [43, Thm 4.2].

Theorem 2 generalises an insight of Denef and Loeser [13, (2.4)] at the heart of their original definition of topological zeta functions of polynomials.

Definition 2 ([38, Def. 5.17]**;** [40, Def. 3.5]**).** In the setting of Theorem 1, we define the *topological subalgebra, submodule, or representation zeta function* $\zeta^{\leqslant}_{\mathsf{A},\mathrm{top}}(s)$, $\zeta_{\Omega\curvearrowright\mathsf{M},\mathrm{top}}(s)$, or $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{G},\mathrm{top}}(s)$, respectively, to be $\mathsf{Z}_{\mathrm{top}}(s) \in \mathbf{Q}(s)$, where Z is defined as in §3.

Up to a simple shift, our definition of topological subalgebra zeta functions is consistent with that of du Sautoy and Loeser [17, §8].

Example 1. Let $\mathfrak{h}$ be the Heisenberg Lie $\mathbf{Z}$-algebra. The subalgebra zeta function of $\mathfrak{h}$ coincides with the subgroup zeta function of the discrete Heisenberg group in (1). Hence, for each prime p, $\zeta^{\leqslant}_{\mathfrak{h}\otimes\mathbf{Z}_p}(s) = W(p,p^{-s})$, where $W(X,Y)$ is given after (1). Thus, the topological subalgebra zeta function of $\mathfrak{h}$ is the constant term of $(1-X^{-1})^3 W(X,X^{-s})$ as a series in $X-1$, i.e.

$$\zeta^{\leqslant}_{\mathfrak{h},\mathrm{top}}(s) = \frac{3s-3}{s(s-1)(2s-2)(2s-3)} = \frac{3}{2s(s-1)(2s-3)}.$$

Observe that the real poles of $\zeta^{\leqslant}_{\mathfrak{h}\otimes\mathbf{Z}_p}(s)$ and $\zeta^{\leqslant}_{\mathfrak{h},\mathrm{top}}(s)$ coincide. While this is not a general phenomenon, Denef and Loeser [13, Thm 2.2] showed that poles of topological zeta functions always give rise to poles of suitable associated local zeta functions. In view of Igusa's Monodromy Conjecture (see [11, §2.3]), this connection between poles of local and topological zeta functions provides one of the key motivations for studying the latter.

Example 1 is misleading in the simplicity of the formula for the topological zeta function and its derivation from knowledge of the associated local zeta functions. Indeed, one of the key features of the method for computing topological zeta functions in §6 is that it does not rely on computations of local zeta functions.

5 Tools from convex geometry

We briefly recall basic notions from convex geometry needed in the following.

5.1 Cones and generating functions

For details on most of the following, see e.g. [4]. A (linear) *half-space* in $\mathbf{R}^n$ is a set of the form $\{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geqslant 0\}$, where $\alpha \in \mathbf{R}^n$ is non-zero and $\langle \cdot, \cdot \rangle$ denotes the usual inner product. If α can be chosen to have rational entries, then the half-space is *rational*. By a *cone* in $\mathbf{R}^n$, we mean a finite intersection of linear half-spaces; note that cones are (convex) polyhedra. If these half-spaces can all be taken to be rational, then we say that the cone is rational. By a *half-open* cone, we mean a set of the form $\mathscr{C}_0 = \mathscr{C} \setminus (\mathscr{C}_1 \cup \cdots \cup \mathscr{C}_r)$, where $\mathscr{C}$ is a cone and each $\mathscr{C}_i$ is a face of $\mathscr{C}$ (i.e. the intersection of $\mathscr{C}$ with a supporting hyperplane). If the $\mathscr{C}_i$ can be chosen to be precisely the faces of $\mathscr{C}$ other than $\mathscr{C}$ itself, then $\mathscr{C}_0$ is a *relatively open cone*. If $\mathscr{C}$ can be chosen to be rational, then we say that $\mathscr{C}_0$ is rational. We say that $\mathscr{C}_0$ is *pointed* if its closure does not contain a non-zero linear subspace. Supposing that $\mathscr{C}_0$ is rational and pointed, it is well-known that the generating function $\sum_{\omega \in \mathscr{C}_0 \cap \mathbf{Z}^n} \boldsymbol{\lambda}^{\omega} \in \mathbf{Q}[\![\lambda_1, \dots, \lambda_n]\!]$ enumerating (integer) lattice points in $\mathscr{C}_0$ is given by a rational function in $\mathbf{Q}(\lambda_1, \dots, \lambda_n)$. The standard proof of rationality proceeds by triangulating the closure of $\mathscr{C}_0$ followed by an application of the inclusion-exclusion principle. This argument does not, in general, lead to a practical algorithm. A more sophisticated method for computing generating functions is “Barvinok’s algorithm” as described by Barvinok and Woods [5]. The implementation of this algorithm as part of LattE [3] plays a vital role in the author’s software Zeta, to be described below.

Half-open cones are convenient for theoretical purposes. However, they appear scarcely in the literature and software usually does not support them directly. Fortunately, as explained in [39, §8.4], we may perform all computations required by the method described below using suitable polyhedra (non-canonically) attached to the half-open cones in question.

5.2 Newton polytopes and initial forms

Most of the following is well-known but the term “balanced” is non-standard; for references, see [38, §4.1].

Let $f \in k[\boldsymbol{X}^{\pm 1}] := k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$. Write $f = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \boldsymbol{X}^\alpha$, where $c_\alpha \in k$ and $c_\alpha = 0$ for almost all $\alpha \in \mathbf{Z}^n$. Let $\mathrm{supp}(f) := \{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$ and define the *Newton polytope* $\mathrm{New}(f)$ of f to be the convex hull of $\mathrm{supp}(f)$ in $\mathbf{R}^n$. Suppose that $f \neq 0$ so that $\mathrm{New}(f) \neq \emptyset$. For $\omega \in \mathbf{R}^n$, let $m(f, \omega) := \min\limits_{\alpha \in \mathrm{supp}(f)} \langle \alpha, \omega \rangle$. We define the *initial form* of f in the direction ω to be

$$\mathrm{in}_\omega(f) := \sum_{\substack{\alpha\in\mathrm{supp}(f),\\ \langle\alpha,\omega\rangle = m(f,\omega)}} c_\alpha \boldsymbol{X}^\alpha.$$

Definition 3 ([39, Def. 5.1(i)]**).** Let $\emptyset \neq \mathscr{M} \subseteq \mathbf{R}^n$ and let $0 \neq f \in k[\boldsymbol{X}^{\pm 1}]$. We say that f is *$\mathscr{M}$-balanced* if $\omega \mapsto \mathrm{in}_\omega(f)$ is constant on $\mathscr{M}$.

Define an equivalence relation $\sim_f$ on $\mathbf{R}^n$ by letting $\omega \sim_f \omega'$ if and only if $\mathrm{in}_\omega(f) = \mathrm{in}_{\omega'}(f)$. Thus, f is $\mathscr{M}$-balanced if and only if $\mathscr{M}$ is contained in one of the equivalence classes of $\sim_f$. We will now recall descriptions of these classes in terms of the Newton polytope of f.

Given a non-empty polytope $\mathcal{P} \subseteq \mathbf{R}^n$ and $\omega \in \mathbf{R}^n$, let $\mathrm{face}_\omega(\mathcal{P})$ be the face of $\mathcal{P}$ consisting of those $\alpha \in \mathcal{P}$ which minimise $\langle \alpha, \omega\rangle$ over $\mathcal{P}$. The (relatively open) *normal cone* of a face $\tau \subseteq \mathcal{P}$ is $\mathrm{N}_\tau(\mathcal{P}) := \{\omega \in \mathbf{R}^n : \mathrm{face}_\omega(\mathcal{P}) = \tau\}$. The equivalence classes of $\sim_f$ from above are precisely the normal cones $\mathrm{N}_\tau(\mathrm{New}(f))$ for faces $\tau \subseteq \mathrm{New}(f)$. In particular, the finite set $\{\mathrm{in}_\omega(f) : \omega \in \mathbf{R}^n\}$ is in natural bijection with the set of faces of $\mathrm{New}(f)$. The following is now obvious.

Lemma 2 ([39, Lem. 5.3]**).** *Let $\emptyset \neq \mathscr{M} \subseteq \mathbf{R}^n$ and $0 \neq f \in k[\boldsymbol{X}^{\pm 1}]$. Then f is $\mathscr{M}$-balanced if and only if there exists a face $\tau \subseteq \mathrm{New}(f)$ with $\mathscr{M} \subseteq \mathrm{N}_\tau(\mathrm{New}(f))$.*

Now suppose that $\boldsymbol{f} = (f_1, \dots, f_r)$ for non-zero $f_1, \dots, f_r \in k[\boldsymbol{X}^{\pm 1}]$. One can show (cf. [38, §3.3]) that the equivalence classes of $\sim_{\boldsymbol{f}}$ defined by letting $\omega \sim_{\boldsymbol{f}} \omega'$ if and only if $\mathrm{in}_\omega(f_i) = \mathrm{in}_{\omega'}(f_i)$ for $i = 1, \dots, r$ are precisely the normal cones associated with faces of $\mathrm{New}(f_1 \cdots f_r)$.

6 A framework for computing zeta functions

In this section, we provide a unified summary of the author's methods for computing generic local and topological zeta functions. For the sake of a more streamlined exposition, we only spell out the case of subalgebra zeta functions.

We begin by recalling the translation step (§6.1) which reduces the computation of local zeta functions to that of computing p-adic integrals. As we will explain in §6.2, these integrals can be encoded in terms of objects that we call "toric data". In §6.3, we introduce the key notions of "balanced" and "regular" toric data. The "simplify-balance-reduce loop" at the heart of our method is discussed in §6.4. Assuming its successful completion, we face different tasks depending on whether we seek to compute (generic) local or topological zeta functions; these tasks are discussed in §6.5.

6.1 p-Adic integration

Let A be an $\mathfrak{o}$-algebra which is free of rank d as an $\mathfrak{o}$-module. By choosing a basis, we identify A and $\mathfrak{o}^d$ as $\mathfrak{o}$-modules. This allows us to parameterise submodules of A using the row spans of upper-triangular $d \times d$ matrices. Building upon work of Grunewald, Segal, and Smith [23, §3], du Sautoy and Grunewald [16, Thm 5.5] observed that those submodules of A which are subalgebras can be characterised in terms of polynomial divisibility conditions in the entries of matrices. We formalise this as in [38, Rem. 2.7(ii)].

Let $R := \mathfrak{o}[\boldsymbol{X}] := \mathfrak{o}[X_{ij} : 1 \leqslant i \leqslant j \leqslant d]$ and $C := [\delta_{i\leqslant j} \cdot X_{ij}] \in \mathrm{Tr}_d(R)$, where $\delta_{i\leqslant j} = 1$ if $i \leqslant j$ and $\delta_{i\leqslant j} = 0$ otherwise. We identify $R^d = \mathsf{A} \otimes_{\mathfrak{o}} R$ and in particular regard R^d as an R-algebra. Let $C_1, \ldots, C_d$ be the rows of C. Let $\mathrm{adj}(C) \in \mathrm{Tr}_d(R)$ be the adjugate matrix of C; hence, over $k(\boldsymbol{X})$, $\mathrm{adj}(C) = \det(C)C^{-1}$.

Henceforth, for $v \in \mathcal{V}_k$, let μ_v denote the additive Haar measure on k_v with $\mu_v(\mathfrak{o}_v) = 1$; we use the same symbol for the product measure on k_v^d and $\mathrm{Tr}_d(k_v)$. Moreover, we let $|\cdot|_v$ denote the usual v-adic absolute value with $|\pi|_v = q_v^{-1}$ for $\pi \in \mathfrak{p}_v \setminus \mathfrak{p}_v^2$. Finally, we write $\|A\|_v := \sup(|a|_v : a \in A)$. The following expresses each $\zeta^{\leqslant}_{\mathsf{A}_v}(s)$ as a "cone integral" in the sense of du Sautoy and Grunewald [16].

Theorem 3 ([16, Thm 5.5]; cf. [23, Prop. 3.1])**.** *Let $\boldsymbol{f} \subseteq \mathfrak{o}[\boldsymbol{X}^{\pm 1}]$ consist of the non-zero entries of all tuples of the form $\det(C)^{-1}(C_m C_n)\,\mathrm{adj}(C)$ for $1 \leqslant m, n \leqslant d$. Then for each $v \in \mathcal{V}_k$,*

$$\zeta^{\leqslant}_{\mathsf{A}_v}(s) = (1 - q_v^{-1})^{-d} \int\limits_{\left\{\boldsymbol{x} \in \mathrm{Tr}_d(\mathfrak{o}_v) : \|\boldsymbol{f}(\boldsymbol{x})\|_v \leqslant 1\right\}} \prod_{i=1}^{d} |x_{ii}|_v^{s-i}\, \mathrm{d}\mu_v(\boldsymbol{x}). \tag{4}$$

We remark that local submodule zeta functions can be similarly expressed in terms of p-adic integrals of the same shape as (4).

Let $\mathbf{G} \leqslant \mathrm{U}_d \otimes k$ be a unipotent algebraic group over k with associated $\mathfrak{o}$-form $\mathsf{G} \leqslant \mathrm{U}_d \otimes \mathfrak{o}$. Stasinski and Voll [49, §2.2.3] expressed $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathfrak{o}_v)}}(s)$, for almost all $v \in \mathcal{V}_k$, in terms of a p-adic integral defined using a fixed set of globally defined polynomials. While the author's framework is flexible enough to accommodate these integrals (see [38, Def. 4.6] and [40, §5.1]), for the sake of simplicity, in the following, we only consider integrals of the form (4).

6.2 Toric data and associated integrals

Henceforth, in addition to k, we fix an "ambient space" of dimension n; in the setting of Theorem 3, our ambient space will be Tr_d so that $n = d(d+1)/2$.

Definition 4 ([39, Def. 3.1]). A *toric datum* is a pair $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$, where $\mathcal{C}_0 \subseteq \mathbf{R}^n_{\geqslant 0}$ is a half-open cone (see §5.1) and $\boldsymbol{f} = (f_1, \dots, f_r)$ is a finite family of non-zero Laurent polynomials $f_i \in k[\boldsymbol{X}^{\pm 1}] := k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$.

Henceforth, we tacitly assume that $\mathcal{C}_0 \neq \emptyset$. We now explain how a toric datum $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ gives rise to p-adic integrals. For $v \in \mathcal{V}_k$ and $\boldsymbol{x} \in k_v^n$, write $v(\boldsymbol{x}) = \big(v(x_1), \dots, v(x_n)\big)$; an elementary but crucial observation is that if $x_1 \cdots x_n \neq 0$ and $\alpha \in \mathbf{Z}^n$, then $v(\boldsymbol{x}^\alpha) = \langle \alpha, v(\boldsymbol{x}) \rangle$. Define $\mathcal{C}_0(\mathfrak{o}_v) := \big\{ \boldsymbol{x} \in \mathfrak{o}_v^n : v(\boldsymbol{x}) \in \mathcal{C}_0 \big\}$.

Definition 5. Let $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ be a toric datum, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$ for $\beta_1, \dots, \beta_m \in \mathbf{N}_0^n$, $v \in \mathcal{V}_k$, and let $s_1, \dots, s_m$ be complex variables. Define

$$\mathsf{Z}_v^{\mathcal{T},\boldsymbol{\beta}}(s_1, \dots, s_m) := \int\limits_{\{\boldsymbol{x} \in \mathcal{C}_0(\mathfrak{o}_v) : \|\boldsymbol{f}(x)\|_v \leqslant 1\}} |\boldsymbol{x}^{\beta_1}|_v^{s_1} \cdots |\boldsymbol{x}^{\beta_m}|_v^{s_m} \, \mathrm{d}\mu_v(\boldsymbol{x}). \tag{5}$$

Thus, the integral in (4) is a univariate specialisation of (5) (with $\mathcal{C}_0 = \mathbf{R}^n_{\geqslant 0}$).

6.3 Balanced and regular toric data

We will now explain how under a suitable regularity hypothesis for a toric datum $\mathcal{T}$, we may construct an explicit (multivariate analogue of) formula (2) for the integrals $\mathsf{Z}_v^{\mathcal{T},\boldsymbol{\beta}}(s_1, \dots, s_m)$. Write $\mathbf{T}^n := \mathrm{Spec}(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ and identify $\mathbf{T}^n(R) = (R^\times)^n$ for any commutative ring R. Let $\bar{k}$ be an algebraic closure of k.

Definition 6 ([39, Def. 5.1(ii), Def. 5.5]). Let $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ be a toric datum with $\boldsymbol{f} = (f_1, \dots, f_r)$ as above.

- $\mathcal{T}$ is *balanced* if f_i is $\mathcal{C}_0$-balanced (see Definition 3) for $i = 1, \dots, r$.
- $\mathcal{T}$ is *regular* if it is balanced and the following holds:
  for each $J \subseteq \{1, \dots, r\}$ and $\boldsymbol{u} \in \mathbf{T}^n(\bar{k})$ with $f_j(\boldsymbol{u}) = 0$ for all $j \in J$, the rank of
  $$\left[\frac{\partial \mathrm{in}_\omega(f_j)(\boldsymbol{u})}{\partial X_i} \right]_{\substack{i=1,\dots,n;\\ j \in J}}$$
  is $\# J$, where $\omega \in \mathcal{C}_0$ is arbitrary (the particular choice of ω being irrelevant).

Using Lemma 2 and the comments following it, we may test if a toric datum $\mathcal{T}$ is balanced. As explained in [39, §5.2], regularity can then be tested using Gröbner bases techniques.

Theorem 4 ([39, Thm 5.8]). *Let $\mathcal{T} = (\mathcal{C}_0; \boldsymbol{f})$ be a regular toric datum, where $\boldsymbol{f} = (f_1, \dots, f_r)$. Let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$ be as in Definition 5. For $J \subseteq \{1, \dots, r\}$, let $V_J^\circ \subseteq \mathbf{T}^n \otimes k$ be the subvariety defined by $f_j = 0$ for $j \in J$ and $f_i \neq 0$ for $i \notin J$. Then there are explicit $W_J \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ such that for almost all $v \in \mathcal{V}_k$,*

$$\mathsf{Z}_v^{\mathcal{T},\boldsymbol{\beta}}(s_1,\dots,s_m) = q_v^{-n} \sum_{J\subseteq\{1,\dots,r\}} \#\bar{V}_J^{\circ}(\mathfrak{K}_v)\cdot(q_v-1)^{\#J}\cdot W_J(q_v,q_v^{-s_1},\dots,q_v^{-s_m}). \quad (6)$$

The W_J in Theorem 4 are given explicitly in the sense that they arise via (explicit) monomial substitutions from generating functions enumerating lattice points inside certain half-open cones $\mathscr{C}_0^J \subseteq \mathscr{C}_0 \times \mathbf{R}^{\#J}$; see [39, §5.5] for details.

Theorem 4 is an algorithmically-minded consequence of [38, Thm 4.10]. The latter theorem provides formulae such as (6) for p-adic integrals of a quite general shape under suitable "non-degeneracy" conditions (closely related to the above concept of regularity for toric data). Such notions of non-degeneracy have their origin in work of Khovanskii [27,28] and others [6,31,52] in toric geometry. They also found numerous applications in the study of Igusa's local zeta function, a close relative of the zeta functions studied here. Indeed, [38, Thm 4.10] was inspired by (and generalises) a result of Denef and Hoornaert [12, Thm 4.2]; another source of inspiration is given by work of Veys and Zúñiga-Galindo [53, §4]. For a more detailed comparison between the author's approach and previous work in the literature, we refer to [38, §4.4].

Much like (2), the formalism for attaching topological zeta functions to families of local ones in §4.2 admits a natural multivariate version; see [38, §5]. However, as a technical inconvenience, we cannot pass directly from (6) to the associated topological zeta function since the Laurent series $(X-1)^{\#J}W_J(X,X^{-s_1},\dots,X^{-s_m})$ in $X-1$ over $\mathbf{Q}(s_1,\dots,s_m)$ typically fail to be *power* series in $X-1$. Fortunately, as explained in [39, §6.4], it turns out that we may rewrite (6) (altering both the varieties and the rational functions involved in the process) in a way that allows us to pass to the associated topological zeta function analogously to Theorem 2. We note that passing from multivariate local zeta functions to topological ones is compatible with suitable univariate specialisations such as the ones used here; see [38, Rem. 5.15].

6.4 The simplify-balance-reduce loop

We now discuss the heart of our method. Starting with an $\mathfrak{o}$-algebra A, we seek to construct a formula (2) for its generic local subalgebra zeta functions. As we have seen, these zeta functions are expressible in terms of p-adic integrals attached to an initial toric datum $\mathcal{T}^0 = (\mathbf{R}^n_{\geqslant 0};\boldsymbol{f}^0)$, where $\boldsymbol{f}^0$ is a set of Laurent polynomials such as the set $\boldsymbol{f}$ in Theorem 3. (The integrand encoded by $\boldsymbol{\beta}$ in Definition 5 is all but insignificant and will be ignored in the following.) Our method is based on several operations applied to toric data as part of a loop.

Overview

At all times of our loop, we maintain a finite collection, $\mathscr{T}$ say, of toric data such that for almost all $v \in \mathcal{V}_k$, the integral (5) associated with our initial toric datum $\mathcal{T}^0$ (essentially the subalgebra zeta function of A_v) is given by the sum of the integrals corresponding to the elements of $\mathscr{T}$; similarly, the topological zeta function associated with $\mathcal{T}^0$ (or, equivalently, the topological subalgebra zeta function of A) will be expressed as a sum of the topological zeta functions attached to the elements of $\mathscr{T}$. Initially, $\mathscr{T}$ only consists of $\mathcal{T}^0$. We repeatedly process those elements of $\mathscr{T}$ that we have not already found to be regular. More precisely, if any such element, $\mathcal{T}$ say, fails to meet certain criteria, then we derive new toric data $\mathcal{T}_1, \dots, \mathcal{T}_N$, say, from $\mathcal{T}$, remove $\mathcal{T}$ from $\mathscr{T}$, insert $\mathcal{T}_1, \dots, \mathcal{T}_N$ into $\mathscr{T}$, and resume processing the elements of$\mathscr{T}$. We now give details on how exactly we process a given toric datum $\mathcal{T} = (\mathscr{C}_0; \boldsymbol{f}) \in \mathscr{T}$.

Simplification

First, we "simplify" $\mathcal{T}$. The key observation is that we may replace $\mathcal{T}$ by any other toric datum if almost all of the associated p-adic integrals remain unchanged. Apart from obvious operations such as removing duplicates or constants from $\boldsymbol{f}$, we are e.g. also free to replace $\boldsymbol{f}$ by another finite generating set of the same $k[\boldsymbol{X}]$-submodule of $k[\boldsymbol{X}^{\pm 1}]$. Moreover, we may remove all Laurent monomials from $\boldsymbol{f}$ for an integrality condition "$|\boldsymbol{x}^\alpha|_v \leqslant 1$" is equivalent to the constraint "$\langle \alpha, v(\boldsymbol{x}) \rangle \geqslant 0$" on $v(\boldsymbol{x})$ which can be encoded by shrinking $\mathscr{C}_0$ accordingly. The precise operations that we carry out are explained in [39, §§7.1–7.2].

Balancing

Suppose that $\mathcal{T}$ has been simplified but that it is not balanced. By considering the non-empty intersections of $\mathscr{C}_0$ with the normal cones of $\mathrm{New}(\prod \boldsymbol{f})$ (see §5.2), we obtain a partition $\mathscr{C}_0 = \bigcup_{i=1}^N \mathscr{C}_0^i$ such that each $\mathcal{T}_i := (\mathscr{C}_0^i; \boldsymbol{f})$ is balanced. We then remove $\mathcal{T}$ from $\mathscr{T}$ and insert $\mathcal{T}_1, \dots, \mathcal{T}_N$.

Reduction

It remains to consider the case that $\mathcal{T}$ is *singular*, i.e. balanced but not regular. The author is unaware of a practically useful method for dealing with these cases in general. Instead, the "reduction step" from [39, §7.3] is an attempt to repair certain specific types of singularity which the author frequently encountered in examples of interest. This method may not lead to immediate improvements and to ensure termination, we impose a bound on the number of subsequent reduction steps. If this number is exceeded, we let our method fail.

Instead of reiterating [39, §7.3], we illustrate the reduction step by discussing the special case that gave rise to the general form. Namely, suppose that $\mathcal{T} = (\mathscr{C}_0; \boldsymbol{f})$ is balanced, where $\boldsymbol{f} = (f_1, \dots, f_r)$ and $r \geqslant 2$. Choose $\omega \in \mathscr{C}_0$. Further suppose that there are $\alpha_1, \alpha_2 \in \mathbf{Z}^n$ and $g \in k[\boldsymbol{X}^{\pm 1}]$ such that $\mathrm{in}_\omega(f_i) = \boldsymbol{X}^{\alpha_i} g$ for $i = 1, 2$; write $h_i = f_i - \boldsymbol{X}^{\alpha_i} g$. We assume that g consists of more than one term (i.e. $\#\mathrm{supp}(g) \geqslant 2$) whence $\mathcal{T}$ is singular. We decompose $\mathscr{C}_0$ into half-open cones $\mathscr{C}_0^{\leqslant}$ and $\mathscr{C}_0^{>}$ defined by

$$\mathscr{C}_0^{\leqslant} := \{\lambda \in \mathscr{C}_0 : \langle \alpha_1, \lambda \rangle \leqslant \langle \alpha_2, \lambda \rangle\} \text{ and}$$
$$\mathscr{C}_0^{>} := \{\lambda \in \mathscr{C}_0 : \langle \alpha_1, \lambda \rangle > \langle \alpha_2, \lambda \rangle\}.$$

Instead of $\mathcal{T}$, we may then consider the two toric data $\mathcal{T}^{\leqslant} := (\mathscr{C}_0^{\leqslant}; \boldsymbol{f})$ and $\mathcal{T}^{>} := (\mathscr{C}_0^{>}; \boldsymbol{f})$. We only consider $\mathcal{T}^{\leqslant}$ in the following, the case of $\mathcal{T}^{>}$ being analogous. We also assume that $\mathscr{C}_0^{\leqslant}$ is non-empty. If $\boldsymbol{x} \in \mathfrak{o}_v^n$ with $\nu(\boldsymbol{x}) \in \mathscr{C}_0^{\leqslant}$, then $v(\boldsymbol{x}^{\alpha_2 - \alpha_1}) \geqslant 0$. It follows that $\mathsf{Z}_v^{\mathcal{T}^{\leqslant}, \boldsymbol{\beta}}(s_1, \dots, s_m)$ remains unaffected if we "remove" one reason for the singularity of $\mathcal{T}^{\leqslant}$, namely the summand "$\boldsymbol{X}^{\alpha_2} g$" of f_2, by replacing f_2 by $f_2' := f_2 - \boldsymbol{X}^{\alpha_2 - \alpha_1} f_1 = h_2 - \boldsymbol{X}^{\alpha_2 - \alpha_1} h_1$. The resulting toric datum may no longer be balanced. We therefore process it using the steps discussed so far in the hope that eventually, all singularities will be successfully removed.

6.5 Processing the pieces

Assuming successful termination of the "simplify-balance-reduce loop", we obtain a formula (2) for $\zeta^{\leqslant}_{\mathsf{A}_v}(s)$ (and almost every $v \in \mathcal{V}_k$) by applying Theorem 4 to each regular toric datum that we constructed. In this formula, the V_i are given as subvarieties of tori $\mathbf{T}^{n_i} \otimes k$ and the $W_i(X,Y)$ are "described" combinatorially (but not yet computed) in terms of generating functions enumerating lattice points inside certain half-open cones. Our next step is to carry out further computations involving the V_i and $W_i(X,Y)$. These computations will depend on whether we seek to compute topological or generic local zeta functions

Topological computations

We first consider the computation of $\zeta^{\leqslant}_{\mathsf{A},\mathrm{top}}(s) \in \mathbf{Q}(s)$. As we mentioned above, by rewriting the formulae obtained using Theorem 4 as in [39, §6.4], we may assume that $W_i(X,Y) \in \mathbf{M}[X,Y]$ for each i in (2); the V_i will still be given as subvarieties (closed ones even) of tori $\mathbf{T}^{n_i} \otimes k$. We are thus left with three steps:

(T1) compute each $\chi(V_i(\mathbf{C}))$,
(T2) compute each $\lfloor W_i(s) \rfloor$, and

(T3) compute $\sum_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i(s) \rfloor$ as a fraction of polynomials from $\mathbf{Q}[s]$.

Regarding (T1), there are general-purpose algorithms for computing Euler characteristics of varieties; see, in particular, work of Aluffi [1]. We do not make any use of these techniques in practice. Instead, we rely on the following two ingredients. First, the Bernstein-Khovanskii-Kushnirenko (BKK) Theorem [28, §3, Thm 2] provides a convex-geometric formula for the topological Euler characteristic of (the complex analytic space associated with) a closed subvariety $f_1 = \cdots = f_m = 0$ of $\mathbf{T}^n \otimes \mathbf{C}$ if $(f_1, \ldots, f_m)$ is non-degenerate in the sense of [27, §2]. Khovanskii's notion of non-degeneracy is closely related to our concept of regularity; see [38, §4.2]. In particular, if the reduction step from §6.4 should not be needed during our computations, then the BKK Theorem can be applied to all varieties that we encounter; cf. [39, Rem. 6.15(ii)]. Secondly, we employ a recursive procedure which seeks to compute topological Euler characteristics associated with closed subvarieties of $\mathbf{T}^n \otimes k$ by decomposing these varieties using subvarieties of lower-dimensional tori. While this procedure is not guaranteed to work in all cases, is has proven to be very useful in practice. Details are given in [39, §6.6] (with some further explanations in [41, §5]).

For (T2), in case $W_i(X,Y)$ is obtained using the method from above, the computation of $\lfloor W_i(s) \rfloor$ is described in [39, §6.5]. An important observation (already used implicitly by Denef and Loeser [13, §5]) is that while $W_i(X,Y)$ arises from a generating function enumerating lattice points inside a half-open cone, $\mathcal{D}_0$ say, $\lfloor W_i(s) \rfloor$ can be written as a sum of rational functions indexed by the cones of maximal dimension in a triangulation of the closure of $\mathcal{D}_0$.

Finally, step (T3) remains. As described in [39, §8.3], we can easily keep track of a common denominator of all $\lfloor W_i(s) \rfloor$ which allows us to recover $\zeta^{\leqslant}_{\mathsf{A},\mathrm{top}}(s)$ using evaluation at random points and polynomial interpolation. This concludes our method for computing topological subalgebra zeta functions.

Generic local computations

Given an associated formula (2) obtained as above, we consider the computation of the generic local subalgebra zeta functions $\zeta^{\leqslant}_{\mathsf{A}_v}(s)$. We make an assumption which is even stronger than uniformity as defined in §3. Namely, we assume that for $i = 1, \ldots, r$, there exists $c_i(X) \in \mathbf{Z}[X]$ such that $\#\bar{V}_i(\mathfrak{K}_v) = c_i(q_v)$ for almost all $v \in \mathcal{V}_k$. The author would like to note that he is not aware of any method for deciding if this assumption is satisfied (or for computing the $c_i(X)$ if it is); for computations in possibly non-uniform settings, see [41, §§5–6,8].

Inspired by steps (T1)–(T3) from above, we proceed as follows:

(L1) attempt to construct each $c_i(X) \in \mathbf{Z}[X]$ (failure being an option),
(L2) compute each $W_i(X,Y)$ as a sum of bivariate rational functions, and

(L3) compute $W(X,Y) \in \mathbf{Q}(X,Y)$ with $\mathsf{Z}^{\leqslant}_{\mathsf{A}_v}(s) = W(q_v, q_v^{-s})$ for almost all $v \in \mathcal{V}_k$.

For (L1), we extend ideas from the computation of Euler characteristics in (T1). We sketch the key ingredients; for details, see [41, §5]. Let $f_1,\dots,f_m \in k[\boldsymbol{X}^{\pm 1}] = k[X_1^{\pm 1},\dots,X_n^{\pm 1}]$ be non-zero. Let $V \subseteq \mathbf{T}^n \otimes k$ be defined by $f_1 = \cdots = f_m = 0$. We seek to find $c(X) \in \mathbf{Z}[X]$ such that $\#\bar{V}(\mathfrak{K}_v) = c(q_v)$ for almost all $v \in \mathcal{V}_k$. This is trivial for $n = 0$. For $n = 1$, the Euclidean algorithm allows us to assume that $m = 1$. We then check if the roots of f_1 lie in k (in which case, we take $c(X)$ to be the number of distinct roots) and abort if it does not. We may thus assume $n > 1$. Similarly to the simplification step in §6.4, we use the fact that we are free to replace the f_i by any collection of Laurent polynomials which generates the same ideal of $k[\boldsymbol{X}^{\pm 1}]$. As one potential reduction of dimensions, we then construct an isomorphism $V \approx_k U \times_k (\mathbf{T}^{n-d} \otimes k)$, where U is a closed subvariety of $\mathbf{T}^d \otimes k$ and $d = \dim(\mathrm{New}(f_1 \cdots f_m))$; see [38, §6.1] and [39, §6.3]. Other potential reductions of dimensions are obtained by trying to solve each $f_i = 0$ for one of the variables as in [41, Lem. 5.1–5.2].

For (L2), we use algorithms due to Barvinok and others [5] for computing and manipulating generating functions associated with polyhedra. Using these methods, each $W_i(X,Y)$ will be expressed as a sum of rational functions. For (L3), similarly to (T3), we write the final sum of (2) over a common denominator. However, due to the frequently large degree of said denominator (in X and Y), at least a naive variation of the approach based on polynomial interpolation from (T3) is often impractical. Instead, after grouping together rational functions based on heuristics (partially inspired by ideas of Woodward [60, §2.5]), we add all numerators over our common denominator. This is usually by far the most computationally involved step of all.

6.6 Zeta

The author's software package Zeta [44] for Sage [51] implements his methods for computing generic local and topological subalgebra, submodule, and representation zeta functions; moreover, Zeta offers basic support for Igusa-type zeta functions associated with polynomials and polynomial mappings (as in [53] but using the author's notion of non-degeneracy instead of [53, Def. 4.1]). Apart from functionality built into Sage, Zeta makes critical use of Singular [9] (polynomial arithmetic, Gröbner bases), Normaliz [8] (triangulations), and LattE [3] (generating functions associated with polyhedra).

7 Highlights of computations using Zeta

The topological subalgebra and representation zeta functions as defined in §4.2 are all invariant under base change in the sense that they only depend on the **C**-

isomorphism class of $\mathsf{A} \otimes_{\mathfrak{o}} \mathbf{C}$ and $\mathsf{G} \otimes_{\mathfrak{o}} \mathbf{C}$, respectively; see [38, Prop. 5.19] and [40, Prop. 4.3]. Apart from the $\mathbf{C}$-isomorphism class of a single 5-dimensional algebra, dubbed Fil_4 by Woodward, the topological subalgebra zeta functions of nilpotent Lie algebras of dimension $\leqslant 5$ can all be derived (via [43, §4]) from previous p-adic calculations. The algebra Fil_4 has a basis $(e_1, \dots, e_5)$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = e_4$, $[e_1, e_4] = e_5$, $[e_2, e_3] = e_5$ and such that all remaining commutators of basis elements (except for those implied by anti-commutativity) are zero. Based on computations using Zeta, the following was first announced in [38, §7.3]:

Theorem 5 ([39, §9.1]**).**

$$\begin{aligned}\zeta_{\mathrm{Fil}_4,\mathrm{top}}(s) = \big(&392031360s^9 - 5741480808s^8 + 37286908278s^7 - \\ &140917681751s^6 + 341501393670s^5 - 550262853249s^4 + \\ &589429290044s^3 - 404678115300s^2 + 161557332768s - \\ &28569052512\big)/\big(3(15s-26)(7s-12)(7s-13)(6s-11)^3 \\ &\quad (5s-8)(5s-9)(4s-7)^2(3s-4)(2s-3)(s-1)s\big).\end{aligned}$$

The seemingly bizarre numbers in the numerator are consistent with the four conjectures stated in [38, §8], two of which we will discuss below. The generic local subalgebra zeta functions associated with Fil_4 remain unknown.

Prior to the following, $\mathfrak{sl}_2(\mathbf{Q})$ was the only example of an insoluble Lie algebra whose associated generic local subalgebra zeta functions had been computed.

Theorem 6 ([41, Thm 9.1]**).** *For almost all primes p, $\zeta^{\leqslant}_{\mathfrak{gl}_2(\mathbf{Z}_p)}(s) = W(p, p^{-s})$, where*

$$\begin{aligned}W(X,Y) = \big(&-X^8Y^{10} - X^8Y^9 - X^7Y^9 - 2X^7Y^8 + X^7Y^7 - X^6Y^8 - X^6Y^7 + 2X^6Y^6 \\ &- 2X^5Y^7 + 2X^5Y^5 - 3X^4Y^6 + 3X^4Y^4 - 2X^3Y^5 + 2X^3Y^3 - 2X^2Y^4 \\ &+ X^2Y^3 + X^2Y^2 - XY^3 + 2XY^2 + XY + Y + 1\big)/\big((1-X^7Y^6) \\ &\quad (1-X^3Y^3)(1-X^2Y^2)^2(1-Y)\big).\end{aligned}$$

Noting that $\mathfrak{gl}_2(\mathbf{Z}_p) \approx \mathfrak{sl}_2(\mathbf{Z}_p) \oplus \mathbf{Z}_p$ for $p \neq 2$, this formula in particular illustrates the generally wild effect of direct sums on subalgebra zeta functions. We note that Theorem 6 is consistent with results of Voll [55, Thm A] and Evseev [21, Thm 3.3].

Other computations of particular interest are that of $\zeta_{\mathrm{U}_d(\mathbf{Z}_p) \curvearrowright \mathbf{Z}_p^d}(s)$ for $d \leqslant 5$ and almost all primes p (see [41, §9.4]); the formula for $d = 5$ fills about three pages. These computations are consistent with functional equations recently established by Voll [58, Thm 5.5] as well as with [42, Prop. 6.1] (which implies that the abscissa of convergence of $\zeta_{\mathrm{U}_d(\mathbf{Z}) \curvearrowright \mathbf{Z}^d}(s)$ is 1 for any $d \geqslant 1$).

Regarding representation zeta functions, extending previous work of others, the author (with the help of Zeta) finished the determination of the generic local representation zeta functions of unipotent algebraic groups of dimension at most 6 over number fields (see [41, §8]); we note that there are infinitely many such groups of dimension 6. The representation zeta functions of $\mathrm{U}_d(\mathbf{Z}_p)$ are only known for $d \leqslant 5$; the case $d = 5$ was settled, for almost all p, using Zeta (see [41, Thm 8.4]).

For comments on limitations of the author's method, see [39, §8.2] and [41, §6.4]. In particular, to the author's knowledge, not a single explicit example of a (local or topological) subalgebra or ideal zeta function associated with a nilpotent Lie algebra of class at least 5 is known. It seems likely that new theoretical insights will be needed to compute such examples. Regarding practical limitations, Zeta can express the generic local subalgebra zeta functions associated with Fil_4 in terms of a sum of bivariate rational functions (thus, in particular, proving uniformity in the sense of §3). However, due to the number and complexity of these rational functions, the author has so far been unable to calculate their sum as a (reduced) fraction of polynomials. The author feels cautiously optimistic that further developments of computational techniques will eventually overcome such obstacles.

8 Conjectures

8.1 Local and topological zeta functions at zero

Every non-trivial local subobject zeta function known to the author has a pole at zero. No explanation of this phenomenon seems to have been provided. Under nilpotency assumptions, much more seems to be true.

Conjecture 1 ([38, Conj. IV]*).* Let A be a nilpotent $\mathfrak{o}$-algebra (associative or Lie, say). Let the underlying $\mathfrak{o}$-module of A be free of rank d. Then for all $v \in \mathcal{V}_k$,

$$\zeta_{\mathsf{A}_v}^{\leqslant}(s) \cdot \left(1 - q_v^{-s}\right) \cdots \left(1 - q_v^{d-1-s}\right)\Big|_{s=0} = 1.$$

Conjecture 1 was first observed by the author in a "topological form" which asserts that $\zeta_{\mathsf{A},\mathrm{top}}^{\leqslant}(s)$ has a simple pole at zero with residue $(-1)^{d-1}/(d-1)!$. (For an example, consider the formula in Theorem 5.) Numerous examples illustrate that Conjecture 1 and its topological form may or may not be satisfied for non-nilpotent examples. The author's "semi-simplification conjecture" [42, Conj. E] disposes of nilpotency assumptions and predicts the exact behaviour of generic local submodule zeta functions $\zeta_{\Omega \curvearrowright \mathsf{M}_v}(s)$ in terms of the Wedderburn decomposition of the associative unital algebra generated by Ω. (The special case $\Omega = \{\omega\}$ of the semi-simplification conjecture follows from [42, Thm 5.1].) It remains an interesting problem to even state a generalisation of Conjecture 1 for possibly non-nilpotent algebras.

8.2 Topological zeta functions at infinity

In contrast to the behaviour at zero in §8.1, the author is not aware of a useful local analogue of the following.

Conjecture 2 ("Degree conjecture"; [38, Conj. I]*).* Let A be an $\mathfrak{o}$-algebra whose underlying $\mathfrak{o}$-module is free of rank d. Then $\zeta^{\leqslant}_{\mathsf{A},\mathrm{top}}(s)$ has degree $-d$ as a rational function in s.

For example, the topological zeta function in Theorem 5 has degree -5, as predicted by Conjecture 2. As explained in [38, §8.1], the degree of a topological zeta function carries valuable information about the associated local zeta functions. We note that [55, Thm A] implies that for almost all $v \in \mathcal{V}_k$, the degree of $\zeta^{\leqslant}_{\mathsf{A}_v}(s)$ as a rational function in q_v^{-s} is $-d$ (cf. [58, §1.3]). A refinement of Conjecture 2 asserts that $s^d \zeta^{\leqslant}_{\mathsf{A},\mathrm{top}}(s)\big|_{s=\infty}$ is a *positive* rational number. Finding an interpretation (even conjectural) of this number remains an interesting open problem.

In contrast to the mysterious case of subobject zeta functions, the author found topological representation zeta functions associated with unipotent groups to always have degree 0; see [40, Cor. 4.7].

Acknowledgements The author thanks Christopher Voll and the anonymous referee for their comments and the Alexander von Humboldt-Foundation for support during the preparation of this article. The work described here was funded by the DFG Priority Programme "Algorithmic and Experimental Methods in Algebra, Geometry and Number Theory" (SPP 1489).

References

[1] P. Aluffi, *Characteristic classes of singular varieties*, Topics in cohomological studies of algebraic varieties, 2005, pp. 1–32.

[2] N. Avni, B. Klopsch, U. Onn, and C. Voll, *Representation zeta functions of compact p-adic analytic groups and arithmetic groups*, Duke Math. J. **162** (2013), no. 1, 111–197.

[3] V. Baldoni, N. Berline, J. A. De Loera, B. Dutra, M. Köppe, S. Moreinis, G. Pinto, M. Vergne, and J. Wu, *A user's guide for LattE integrale v1.7.3*, 2015. Software package. LattE is available at `http://www.math.ucdavis.edu/~latte/`.

[4] A. Barvinok, *Integer points in polyhedra*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[5] A. Barvinok and K. Woods, *Short rational generating functions for lattice point problems*, J. Amer. Math. Soc. **16** (2003), no. 4, 957–979 (electronic).

[6] D. N. Bernstein, A. G. Kushnirenko, and A. G. Khovanskii, *Newton polyhedra (Russian)*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 201–202.

[7] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302.

[8] W. Bruns, B. Ichim, T. Römer, and C. Söger, *Normaliz 3.1.2. Algorithms for rational cones and affine monoids.*, 2016. Available from `http://www.math.uos.de/normaliz/`.

[9] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, SINGULAR *4-1-0 — A computer algebra system for polynomial computations*, 2016. Available from `http://www.singular.uni-kl.de/`.

[10] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice ıt Corps de classes local par Michiel Hazewinkel.

[11] J. Denef, *Report on Igusa's local zeta function*, Astérisque **201-203** (1991), Exp. No. 741, 359–386 (1992). Séminaire Bourbaki, Vol. 1990/91.

[12] J. Denef and K. Hoornaert, *Newton polyhedra and Igusa's local zeta function*, J. Number Theory **89** (2001), no. 1, 31–64.

[13] J. Denef and F. Loeser, *Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques*, J. Amer. Math. Soc. **5** (1992), no. 4, 705–720.

[14] M. P. F. du Sautoy, *The zeta function of* $\mathfrak{sl}_2(\mathbb{Z})$, Forum Math. **12** (2000), no. 2, 197–221.

[15] ———, *Zeta functions of groups: the quest for order versus the flight from ennui*, Groups St. Andrews 2001 in Oxford. Vol. I, 2003, pp. 150–189.

[16] M. P. F. du Sautoy and F. J. Grunewald, *Analytic properties of zeta functions and subgroup growth*, Ann. of Math. (2) **152** (2000), no. 3, 793–833.

[17] M. P. F. du Sautoy and F. Loeser, *Motivic zeta functions of infinite-dimensional Lie algebras*, Selecta Math. (N.S.) **10** (2004), no. 2, 253–303.

[18] M. P. F. du Sautoy and G. Taylor, *The zeta function of* $\mathfrak{sl}_2$ *and resolution of singularities*, Math. Proc. Cambridge Philos. Soc. **132** (2002), no. 1, 57–73.
[19] M. P. F. du Sautoy and L. Woodward, *Zeta functions of groups and rings*, Lecture Notes in Mathematics, vol. 1925, Springer-Verlag, Berlin, 2008.
[20] D. H. Dung and C. Voll, *Uniform analytic properties of representation zeta functions of finitely generated nilpotent groups*, Trans. Amer. Math. Soc. **369** (2017), no. 9, 6327–6349.
[21] A. Evseev, *Reduced zeta functions of Lie algebras*, J. Reine Angew. Math. **633** (2009), 197–211.
[22] S. Ezzat, *Representation growth of finitely generated torsion-free nilpotent groups: Methods and examples*, Ph.D. Thesis, 2012. See `http://hdl.handle.net/10092/7235`.
[23] F. J. Grunewald, D. Segal, and G. C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988), no. 1, 185–223.
[24] E. Hrushovski, B. Martin, S. Rideau, and R. Cluckers, *Definable equivalence relations and zeta functions of groups*, To appear in J. Eur. Math. Soc. (JEMS) (2017). arXiv:math/0701011.
[25] I. Ilani, *Zeta functions related to the group* $\mathrm{SL}_2(\mathbb{Z}_p)$, Israel J. Math. **109** (1999), 157–172.
[26] A. Jaikin-Zapirain, *Zeta function of representations of compact p-adic analytic groups*, J. Amer. Math. Soc. **19** (2006), no. 1, 91–118 (electronic).
[27] A. G. Khovanskii, *Newton polyhedra, and toroidal varieties*, Funkcional. Anal. i Priložen. **11** (1977), no. 4, 56–64, 96.
[28] ______, *Newton polyhedra, and the genus of complete intersections*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 1, 51–61.
[29] B. Klopsch, *Representation growth and representation zeta functions of groups*, Note Mat. **33** (2013), no. 1, 107–120.
[30] B. Klopsch and C. Voll, *Zeta functions of three-dimensional p-adic Lie algebras*, Math. Z. **263** (2009), no. 1, 195–210.
[31] A. G. Kushnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), no. 1, 1–31.
[32] M. Larsen and A. Lubotzky, *Representation growth of linear groups*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 2, 351–390.
[33] S. Lee and C. Voll, *Enumerating graded ideals in graded rings associated to free nilpotent Lie rings (preprint)* (2016). arXiv:1606.04515.
[34] M. W. Liebeck, *Probabilistic and asymptotic aspects of finite simple groups*, Probabilistic group theory, combinatorics, and computing, 2013, pp. 1–34.
[35] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336, xi+117.
[36] J. Nicaise, *An introduction to p-adic and motivic zeta functions and the monodromy conjecture*, Algebraic and analytic aspects of zeta functions and L-functions, 2010, pp. 141–166.
[37] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. **399** (1989), 137–172.
[38] T. Rossmann, *Computing topological zeta functions of groups, algebras, and modules, I*, Proc. Lond. Math. Soc. (3) **110** (2015), no. 5, 1099–1134.
[39] ______, *Computing topological zeta functions of groups, algebras, and modules, II*, J. Algebra **444** (2015), 567–605.
[40] ______, *Topological representation zeta functions of unipotent groups*, J. Algebra **448** (2016), 210–237.
[41] ______, *Computing local zeta functions of groups, algebras, and modules*, To appear in Trans. Amer. Math. Soc. (2017). arXiv:1602.00919.
[42] ______, *Enumerating submodules invariant under an endomorphism*, Math. Ann. **368** (2017), no. 1-2, 391–417.
[43] ______, *Stability results for local zeta functions of groups algebras, and modules*, Math. Proc. Cambridge Philos. Soc. (2017), 1–10. DOI:10.1017/S0305004117000585.

[44] ———, Zeta, *version 0.3.2*, 2017. See http://www.math.uni-bielefeld.de/~rossmann/Zeta/.
[45] J.-P. Serre, *Lectures on $N_X(p)$*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 11, CRC Press, Boca Raton, FL, 2012.
[46] A. Shalev, *Applications of some zeta functions in group theory*, Zeta functions in algebra and geometry, 2012, pp. 331–344.
[47] R. Snocken, *Zeta functions of groups and rings*, Ph.D. Thesis, 2012. See http://eprints.soton.ac.uk/id/eprint/372833.
[48] L. Solomon, *Zeta functions and integral representation theory*, Advances in Math. **26** (1977), no. 3, 306–326.
[49] A. Stasinski and C. Voll, *Representation zeta functions of nilpotent groups and generating functions for Weyl groups of type B*, Amer. J. Math. **136** (2014), no. 2, 501–550.
[50] A. Stasinski and C. Voll, *Representation zeta functions of some nilpotent groups associated to prehomogeneous vector spaces*, Forum Math. **29** (2017), no. 3, 717–734.
[51] W. A. Stein et al., *Sage Mathematics Software (Version 7.4)*, The Sage Development Team, 2016. Available from http://www.sagemath.org/.
[52] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.
[53] W. Veys and W. A. Zúñiga-Galindo, *Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra*, Trans. Amer. Math. Soc. **360** (2008), no. 4, 2205–2227.
[54] O. Villamayor, *Constructiveness of Hironaka's resolution*, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 1, 1–32.
[55] C. Voll, *Functional equations for zeta functions of groups and rings*, Ann. of Math. (2) **172** (2010), no. 2, 1181–1218.
[56] ———, *A newcomer's guide to zeta functions of groups and rings*, Lectures on profinite topics in group theory, 2011, pp. 99–144.
[57] ———, *Zeta functions of groups and rings—recent developments*, Groups St Andrews 2013, 2015, pp. 469–492.
[58] ———, *Local functional equations for submodule zeta functions associated to nilpotent algebras of endomorphisms*, To appear in Int. Math. Res. Not. IMRN (2017). arXiv:1602.07025.
[59] E. Witten, *On quantum gauge theories in two dimensions*, Comm. Math. Phys. **141** (1991), no. 1, 153–209.
[60] L. Woodward, *Zeta functions of groups: computer calculations and functional equations*, Ph.D. Thesis, 2005.