Periodicities for graphs of p-groups beyond coclass

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Abstract. We use computational methods to investigate periodic patterns in the graphs $\mathcal{G}(p,(d,w,o))$ associated with the p-groups of rank d, width w, and obliquity o. In the smallest interesting case $\mathcal{G}(p,(3,2,0))$ we obtain a conjectural description of this graph for all $p \geq 3$; in particular, we conjecture that this graph is virtually periodic for all $p \geq 3$. We also investigate other related infinite graphs.

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1. Introduction

Which invariants are useful in the classification of p-groups?

The *order* has been considered in many publications, going back to the beginnings of abstract group theory in the 19th century; see [**1**] for a history. Nowadays, the p-groups of order dividing 2^9 (see [**5**]) and p^7 (see [**16**]) are available, but a full classification of the groups of order p^n in general still seems to be out of reach. An important step towards a full classification would be a proof of the famous PORC conjecture [**7**] which asserts that for fixed n, the number $f(p)$ of p-groups of order p^n is a polynomial on residue classes.

Leedham-Green and Newman [**14**] suggested using the *coclass* to classify p-groups. Recall that the coclass of a finite p-group G of order p^n and nilpotency class $\mathrm{cl}(G)$ is defined as $\mathrm{cc}(G) = n - \mathrm{cl}(G)$. A first and fundamental idea in classifying all p-groups of a given coclass r is to visualize them in a graph $\mathcal{G}(p,r)$: the vertices of this graph correspond to the isomorphism types of p-groups of coclass r and there is a directed edge $G \to H$ if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$ holds, where $\gamma_i(H)$ denotes the ith term of the lower central series of H. The classification of all p-groups of coclass r thus translates to an investigation of the infinite graph $\mathcal{G}(p,r)$.

Coclass theory has become a rich and interesting research field in group theory. A highlight in this theory was the complete proof of the coclass-conjectures [**14**] by Shalev [**18**] and Leedham-Green [**11**]. We refer to the book by Leedham-Green and McKay [**13**] for background and details. Nowadays, the fundamental aim in coclass theory is to prove that every graph $\mathcal{G}(p,r)$ can be constructed from a finite subgraph using certain periodic patterns. This has been proved for $p = 2$ in [**3**] and

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[**4**], but is still open for odd primes. A central problem in the odd prime case is that the graphs $\mathcal{G}(p,r)$ are usually rather thick and thus are often difficult to investigate in detail. As a consequence, only very little detailed experimental evidence on the structure of these graphs is available and explicit conjectures on the nature of any useful periodic patterns are vague at present.

Leedham-Green thus suggested to try other invariants with a similar approach as in coclass theory with the hope of obtaining graphs which have all the nice features of the graphs $\mathcal{G}(p,r)$, but are thinner and thus easier to understand. In particular, Leedham-Green initiated the classification of p-groups by *rank, width and obliquity*; see Chapter 12 of [**13**] for a discussion. We briefly recall the definitions of these invariants: for any finite or infinite pro-p-group G and a closed subgroup H of G, let $[G:H]_p$ denote the p-logarithm of the index $[G:H]$; further let $\mathrm{d}(G) = [G:\Phi(G)]_p$ be the cardinality of a minimal (topological) generating set of G, and let $\mu_i(G)$ denote the intersection of all closed normal subgroups of G which are not properly contained in $\gamma_i(G)$. Then we define for a pro-p-group G:

- its *rank* $\mathrm{r}(G) = \sup\{\mathrm{d}(U) \mid U \text{ a closed subgroup of } G\}$,
- its *width* $\mathrm{w}(G) = \sup\{[\gamma_i(G) : \gamma_{i+1}(G)]_p \mid i \in \mathbb{N}\}$, and
- its *obliquity* $\mathrm{o}(G) = \sup\{[\gamma_i(G) : \mu_i(G)]_p \mid i \in \mathbb{N}\}$.

The obliquity of a group determines how restricted its lattice of normal subgroups is. In particular, in a group of obliquity 0 every normal subgroup lies between two consecutive terms of the lower central series.

Let $\tau(G)$ denote the triple $(\mathrm{r}(G), \mathrm{w}(G), \mathrm{o}(G))$ and define the graph $\mathcal{G}(p,(d,w,o))$ similar to the coclass graphs: the vertices of this graph correspond to the isomorphism types of finite p-groups G with $\tau(G) = (d,w,o)$ and there is a directed edge $G \to H$ if $G \cong H/\gamma_{\mathrm{cl}(H)}(H)$ holds. The classification of all p-groups G with $\tau(G) = (d,w,o)$ now translates to understanding the (usually) infinite graph $\mathcal{G}(p,(d,w,o))$.

In this paper we discuss how computational tools can be used to investigate the graphs $\mathcal{G}(p,(d,w,o))$ and we exhibit experimental results for some small and interesting cases. Thus, we give a conjectural description of the graph $\mathcal{G}(p,(3,2,0))$ for $p > 2$ based on our experimental data. It suggests that $\mathcal{G}(p,(3,2,0))$ can be constructed from a finite subgraph using certain periodic patterns and hence $\mathcal{G}(p,(3,2,0))$ seems to have the nice features displayed by the coclass graphs $\mathcal{G}(2,r)$ and, moreover, it is a rather thin graph which can be easily exhibited.

An interesting family of infinite pro-p-groups G with finite $\tau(G)$ are the Sylow pro-p-subgroups of $\mathrm{Aut}(L)$ for simple Lie algebras L of the type $L = s\ell_n(K)$ for $p \geq 3$, where $K/\mathbb{Q}_p$ is a finite extension. The lower central series quotients of such a group G define an infinite path through the graph $\mathcal{G}(p,\tau(G))$. We show how our computational tools can be used to investigate these infinite paths together with certain branches associated with them. Our experiments with these infinite trees indicate that they also exhibit periodic patterns of the same type as $\mathcal{G}(p,(3,2,0))$.

Throughout this paper we assume that p is an odd prime.

2. Preliminaries

There is a correspondence between the infinite paths in $\mathcal{G}(p,(d,w,o))$ and the isomorphism types of infinite pro-p-groups G with $\tau(G) = (d,w,o)$. Hence a first aim in understanding $\mathcal{G}(p,(d,w,o))$ is a classification of the infinite pro-p-groups G

with $\tau(G) = (d, w, o)$. In this section, we recall some basic facts about these groups. Recall that a pro-p-group is *just infinite* if it is infinite but every non-trivial closed normal subgroup has finite index.

LEMMA 2.1. *Let G be an infinite pro-p-group of finite rank, finite width, and finite obliquity. Then G is p-adic analytic and just infinite.*

PROOF. A pro-p-group of finite rank is p-adic analytic by [**2**], Corollary 8.33. An infinite pro-p-group of finite width and finite obliquity is just infinite, see [**10**], p. 3. □

If G is a pro-p-group of finite width, then $G/\gamma_i(G)$ is finite for all $i \in \mathbb{N}$. If, moreover, $\tau(G)$ is finite, then there exists a $k \in \mathbb{N}$ with $\tau(G/\gamma_i(G)) = \tau(G)$ for all $i \geq k$ and the groups $G/\gamma_i(G)$ for $i \geq k$ thus define an infinite path through the graph $\mathcal{G}(p, \tau(G))$.

2.1. The solvable case. Let G be an infinite solvable pro-p-group of finite rank, width, and obliquity. Then G is an irreducible p-adic space group, see for example [**19**], Lemma 8.1. This means that G is an extension of a free p-adic module $\mathbb{Z}_p^d$ for some $d \in \mathbb{N}$ by a finite p-group P which acts faithfully on $\mathbb{Z}_p^d$ and irreducibly on $\mathbb{Q}_p^d$.

The possible dimensions d and point groups P are well-understood, see for example [**13**]. Excluding the trivial case $d = 1$, irreducible finite p-subgroups of $\mathrm{GL}_d(\mathbb{Z}_p)$ only exist for dimensions d of the form $d = p^{r-1}(p-1)$ for some $r \in \mathbb{N}$. For such d, the group $\mathrm{GL}_d(\mathbb{Z}_p)$ has precisely p^{r-1} conjugacy classes of Sylow p-subgroups with representatives $W_1, \ldots, W_{p^{r-1}}$, say. These groups $W_1, \ldots, W_{p^{r-1}}$ are all conjugate in $\mathrm{GL}_d(\mathbb{Q}_p)$ to an r-fold iterated wreath product $\mathrm{C}_p \wr \cdots \wr \mathrm{C}_p$ of cyclic groups of order p.

It is well-known that each p-adic space group can be embedded as a subgroup of finite index into a split space group; that is, a space group which is a split extension of $\mathbb{Z}_p^d$ by P, see for example [**13**], Lemma 10.4.1. This implies the following embedding theorem.

THEOREM 2.2. *Let G be an irreducible p-adic space group of dimension $d = p^{r-1}(p-1)$. Then G embeds as subgroup of finite index into $\mathbb{Z}_p^d \rtimes W_i$ for some $i \in \{1, \ldots, p^{r-1}\}$.*

A simple example of an infinite solvable pro-p-group of finite rank, width, and obliquity is the group $\mathbb{Z}_p^{p-1} \rtimes \mathrm{C}_p$. This group has rank $p-1$, width 2, and obliquity 0. It also has finite coclass; in fact, it is the unique infinite pro-p-group of coclass 1.

2.2. The insolvable case. The insolvable infinite pro-p-groups of finite rank, width, and obliquity are discussed in detail in [**10**]. Here we only recall some of their features. The following theorem yields a characterization of these groups, see Lemma 2.1 and [**10**], Section IIId.

THEOREM 2.3. *Let G be an insolvable infinite pro-p-group. Then G has finite rank, width, and obliquity if and only if G is p-adic analytic and just infinite.*

Every pro-p-group of finite rank has an associated Lie algebra, see for example [**2**], Section 9.5. In the case of an insolvable infinite pro-p-group of finite rank, width and obliquity, this Lie algebra is homogeneous of the form $S^{p^k} = S \oplus \cdots \oplus S$ (p^k summands) for a simple finite-dimensional Lie algebra S over $\mathbb{Q}_p$ and some

$k \geq 0$, see [**10**], Proposition III.6. The following theorem shows how such a pro-p-group can be recovered from its associated Lie algebra; we refer to [**10**], Proposition III.9 and Corollary III.10 for details and background.

THEOREM 2.4. *Let G be an insolvable infinite pro-p-group of finite rank, width, and obliquity, and let L be its associated Lie algebra over $\mathbb{Q}_p$. Then G embeds as a subgroup of finite index into a Sylow pro-p-subgroup of* $\mathrm{Aut}(L)$.

If $L = S^{p^k}$ for a simple Lie algebra S, then $\mathrm{Aut}(L) = \mathrm{Aut}(S) \wr \mathrm{Sym}(p^k)$ and thus the Sylow pro-p-subgroups of $\mathrm{Aut}(L)$ can be determined from the Sylow pro-p-subgroups of $\mathrm{Aut}(S)$ and $\mathrm{Sym}(p^k)$. The Sylow p-subgroups of $\mathrm{Sym}(p^k)$ are well-understood and can be constructed readily. It is known from [**10**], Lemma III.16 that the Sylow pro-p-subgroups of $\mathrm{Aut}(L)$ are all conjugate.

Theorem 2.4 suggests an approach for constructing the insolvable infinite pro-p-groups of finite rank, width and obliquity: first classify the finite-dimensional simple Lie algebras over $\mathbb{Q}_p$, then determine the Sylow pro-p-subgroups of their automorphism groups, and finally list the relevant subgroups of finite index in the relevant wreath products based on these Sylow pro-p-subgroups. None of the steps in this approach is straightforward in practice. For example, in [**10**] the Sylow pro-p-subgroups of the homogeneous Lie algebras of dimension at most 14 are determined and this proved to be highly non-trivial.

Examples of simple Lie algebras over $\mathbb{Q}_p$ are the linear Lie algebras $\mathfrak{sl}_d(K)$ consisting of all $d \times d$ matrices with trace 0 over an extension K of $\mathbb{Q}_p$. If K has degree m over $\mathbb{Q}_p$, then $\mathfrak{sl}_d(K)$ has dimension $m(d^2-1)$ over $\mathbb{Q}_p$. Note that there are only finitely many field extensions K of any given degree over $\mathbb{Q}_p$. The group $\mathrm{Aut}(\mathfrak{sl}_d(K))$ of automorphisms over $\mathbb{Q}_p$ is $(\mathrm{PGL}_d(K) \rtimes D) \rtimes \mathrm{Gal}(K/\mathbb{Q}_p)$, where D is the group of so-called diagram automorphisms. The Sylow pro-p-subgroup of $\mathrm{Aut}(\mathfrak{sl}_d(K))$ is explicitly determined in [**10**], Lemma XI.4.

2.3. The pro-p-groups of rank 3, width 2, and obliquity 0. A complete classification of the infinite pro-p-groups of rank 3, width 2, and obliquity 0 up to isomorphism is given in [**13**], Theorem 12.2.3: for every $p > 2$, there are two groups of this type (up to isomorphism). We briefly recall their description.

The first group is the Sylow pro-p-subgroup of $\mathrm{Aut}(L)$, where L is the simple 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{Q}_p)$ of 2×2-matrices with trace 0 over $\mathbb{Q}_p$. As shown in [**13**], this group can be identified with the subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ consisting of the matrices which are upper unitriangular modulo p.

There is exactly one other isomorphism type of simple Lie algebras of dimension 3 over $\mathbb{Q}_p$. To construct a representative, M say, for this isomorphism type, let $\mathbb{Q}_p(a)$ be the unramified extension of $\mathbb{Q}_p$ of degree 2, where a has multiplicative order p^2-1. Let π be the automorphism of $\mathbb{Q}_p(a)$ with $a^\pi = a^p$ and define $K = \mathbb{Q}_p(a,\pi)$. Then K is a division algebra of degree 4 over $\mathbb{Q}_p$. Commutation in K defines a Lie algebra H and $M = [H,H]$ is the desired simple Lie algebra.

The second infinite pro-p-group of rank 3, width 2, and obliquity 0 is the Sylow pro-p-subgroup of $\mathrm{Aut}(M)$. As shown in [**13**], this group can be identified with the central quotient $U/Z(U)$, where U is the group of 1-units of $\mathbb{Z}_p(a,\pi)$.

3. The graphs $\mathcal{G}(p,(d,w,o))$

For a group G we write $G_i = G/\gamma_i(G)$ for $i \in \mathbb{N}$. If G is an infinite pro-p-group with $\tau(G) = (d,w,o)$, then there exists an integer k such that the quotients

G_i for $i \geq k$ form an infinite path through $\mathcal{G}(p,(d,w,o))$. Conversely, the inverse limit of the groups on an infinite path in $\mathcal{G}(p,(d,w,o))$ is an infinite pro-p-group G with $\tau(G) = (d,w,o)$. Hence the infinite pro-p-groups G with $\tau(G) = (d,w,o)$ parametrize the graph $\mathcal{G}(p,(d,w,o))$ in this sense. Our aim is to investigate the graphs $\mathcal{G}(p,(d,w,o))$ based on this parametrization.

3.1. Descendant trees and τ-trees. For a finite p-group H we denote by $\mathcal{T}(H)$ the *descendant tree* of H: the vertices of $\mathcal{T}(H)$ correspond to the isomorphism types of finite p-groups K such that $K_{\mathrm{cl}(H)+1} \cong H$ and there is an edge $K \to L$ if $L_{\mathrm{cl}(L)} \cong K$. The groups in $\mathcal{T}(H)$ are called *descendants* of H. An *immediate descendant* of H is a descendant of H of class $\mathrm{cl}(H)+1$.

Let G be an infinite pro-p-group with $\tau(G) = (d,w,o)$ and let i be minimal such that $\tau(G_i) = (d,w,o)$ and $G_i \ncong H_i$ for any infinite pro-p-group $H \ncong G$ with $\tau(H) = (d,w,o)$. Since there are only finitely many possible isomorphism types for H, see [**12**], p. 72, such an i exists. The *τ-tree* $\mathcal{T}(G,(d,w,o))$ is defined as the intersection of $\mathcal{T}(G_i)$ with $\mathcal{G}(p,(d,w,o))$; it thus consists of all descendants K of G_i with $\tau(K) = (d,w,o)$. Note that $G_i \to G_{i+1} \to \ldots$ is the unique infinite path in $\mathcal{T}(G,(d,w,o))$ starting at the root G_i of this tree.

3.2. Virtual periodicity of trees. Let $\mathcal{T}$ be a tree with root R having a unique infinite path $R = R_1 \to R_2 \to \ldots$. For each $i \in \mathbb{N}$ let $\mathcal{B}_{R_i}(\mathcal{T})$ be the subtree of $\mathcal{T}$ consisting of the descendants of R_i which are not descendants of R_{i+1}. Then $\mathcal{B}_{R_i}(\mathcal{T})$ is a *branch* of $\mathcal{T}$. We say that $\mathcal{T}$ is *virtually periodic* if there exist l and d such that

$$\mathcal{B}_{R_i}(\mathcal{T}) \cong \mathcal{B}_{R_{i+d}}(\mathcal{T})$$

for every $i \geq l$. We then call the least possible value of such a d the *period* and the corresponding least possible value of l the *defect* of $\mathcal{T}$. Note that a virtually periodic tree $\mathcal{T}$ with defect l and period d can be constructed from its first $l+d-1$ branches $\mathcal{B}_{R_1}(\mathcal{T}), \ldots, \mathcal{B}_{R_{l+d-1}}(\mathcal{T})$.

3.3. Virtual periodicity of $\mathcal{G}(p,(d,w,o))$. We say the graph $\mathcal{G}(p,(d,w,o))$ is *virtually periodic* if all but finitely many groups of $\mathcal{G}(p,(d,w,o))$ are contained in a τ-tree of $\mathcal{G}(p,(d,w,o))$ and if every τ-tree of $\mathcal{G}(p,(d,w,o))$ is virtually periodic. If $\mathcal{G}(p,(d,w,o))$ is virtually periodic, then it can be constructed from a finite subgraph and this would furnish a classification of the p-groups with rank d, width w, and obliquity o. However, the following interesting question is wide open at current.

Question: *For which primes p and which (d,w,o) is $\mathcal{G}(p,(d,w,o))$ virtually periodic?*

It is hoped that for every prime p and every (d,w,o) all but finitely many groups of $\mathcal{G}(p,(d,w,o))$ are contained in a τ-tree of this graph.

It can be deduced from [**13**], Proposition 3.1.2 and Exercise 3.3(3) that if $p > 2$ and P is a finite p-group of maximal class and order at least p^{p+1}, then $\tau(P) = (p-1,2,0)$ holds. Consequently, all but finitely many groups from the coclass graph $\mathcal{G}(p,1)$ are contained in the graph $\mathcal{G}(p,(p-1,2,0))$. Since it is known that the (unique) maximal infinite subtree of $\mathcal{G}(p,1)$ is not virtually periodic in the sense of Section 3.2 for $p > 3$, it follows that not every graph $\mathcal{G}(p,(d,w,o))$ is virtually periodic.

We conjecture below that the graph $\mathcal{G}(p,(3,2,0))$ is virtually periodic for all primes $p \geq 3$.

4. Computational methods

Many computational methods work with the lower p-series rather than the lower central series: the *lower p-series* of a finite p-group H is defined by $\lambda_1(H) = H$ and $\lambda_{i+1}(H) = [\lambda_i(H), H]\lambda_i(H)^p$ for $i \geq 1$. The length of this series is the *p-class* of H. A finite p-group K is an *immediate p-descendant* of H if $K/\lambda_{d+1}(K) \cong H$, where d is the p-class of H, and if, in addition, the p-class of K is $d+1$.

The ANUPQ [**15**] program allows the determination of all immediate p-descendants of a given non-trivial finite p-group H up to isomorphism. We want to use this program to determine the immediate descendants of H. For this purpose we investigate under which circumstances the p-descendants coincide with the descendants. We call a finite p-group H *stable* if $\gamma_i(H) = \lambda_i(H)$ holds for all i.

Lemma 4.1. *Suppose that H is non-abelian and stable. Then E is an immediate descendant of H if and only if it is an immediate p-descendant of H.*

Proof. It suffices to show that either condition on E implies that E is stable. Thus let E be an immediate descendant of H. As $E/\gamma_2(E) = E/\lambda_2(E)$, it follows from [**8**], Kapitel III, Satz 2.13b that every lower central factor of E has exponent dividing p. Hence E is stable. Conversely, let E be an immediate p-descendant of H. Then $E/\lambda_2(E) \cong H/\lambda_2(H) = H/H' \cong E/\lambda_{c+1}(E)E'$ and hence $\lambda_2(E) = \lambda_{c+1}(E)E'$. Therefore, $\lambda_2(E)/E' = \lambda_{c+1}(E)E'/E' \cong \lambda_{c+1}(E)/\lambda_{c+1}(E) \cap E'$ has exponent dividing p, whence $\lambda_{c+1}(E) \leq \lambda_3(E) \leq E'$. Thus, $E' = \lambda_2(E)$ and E is stable. □

In this manner the ANUPQ program allows us to construct finite subtrees of a descendant tree $\mathcal{T}(H)$ for a non-abelian and stable p-group H. The resulting descendants are all described in terms of power-commutator (pc) presentations. Such presentations allow effective computations with the groups they define and we can further investigate the resulting groups using the computer algebra system GAP [**20**].

The width of a group defined by a pc presentation can be read off quite readily: we need to determine the lower central series of the groups under consideration. Even simpler, if the group in question is a descendant determined by ANUPQ, then its lower central series and thus its width can be read off directly.

The rank and the obliquity of a group defined by a pc presentation can be computed, but the available algorithms for this purpose are significantly less effective than those for the computation of the width. We use the formula from [**10**], p. 74, to determine the obliquity of a finite p-group H. That is, we use the formula

$$\mu_i(H) = \mu_{i-1}(H) \cap \gamma_i(H) \cap \bigcap \{N \triangleleft H \mid N \not\leq \gamma_i(H) \text{ and } N \leq \gamma_{i-1}(H)\}$$

and then obtain that the obliquity of H is given by

$$\mathrm{o}(H) = \max_i \log_p [\gamma_i(H) : \mu_i(H)].$$

We determine the rank of a finite p-group H by computing a representative for every conjugacy class of subgroups of H and determining the maximum of their minimal generator numbers.

We combine these methods to the following approach for determining finite parts of the τ-tree for an infinite pro-p-group G which has finite rank, width and obliquity and a stable quotient $G/\gamma_2(G)$. Note that the latter condition implies that all lower central quotients $G_i = G/\gamma_i(G)$ are stable. First we choose i large enough such that G_i satisfies $\tau(G_i) = \tau(G) = (d, w, o)$ and $G_i \not\cong H_i$ for any infinite pro-p-group H with $H \not\cong G$ and $\tau(H) = (d, w, o)$. Then we apply the ANUPQ program to start generating the descendant tree of G_i. For every obtained descendant K, we check whether it satisfies $\tau(K) = (d, w, o)$ and if this is not the case, then we discard K. The remaining descendants are groups in $\mathcal{T}(G, (d, w, o))$.

Implementations of our main methods can be found in the Fwtree package [**6**] for the computer algebra system GAP.

5. The graph $\mathcal{G}(p, (3, 2, 0))$

We have used the algorithms of Section 4 to investigate the graph $\mathcal{G}(p, (3, 2, 0))$ for various primes $p > 2$. This section summarizes our results.

Recall that for every prime $p > 2$ the graph $\mathcal{G}(p, (3, 2, 0))$ contains two τ-trees; these correspond to the two isomorphism types of infinite pro-p-groups of rank 3, width 2, and obliquity 0. Let G and H denote representatives for these isomorphism types, where G corresponds to the Lie algebra $s\ell_2(\mathbb{Q}_p)$.

The following lemma reduces the investigation of $\mathcal{G}(p, (3, 2, 0))$ to an investigation of its τ-trees.

Lemma 5.1. *If $p > 2$, then almost all groups in $\mathcal{G}(p, (3, 2, 0))$ are contained in a τ-tree.*

Proof. For $p \geq 5$, this is a consequence of [**13**], Theorem 12.2.15. For $p = 3$, we can explicitly determine the groups not contained in the τ-trees. If P is a 3-group with $\tau(P) = (3, 2, 0)$, then its class-2 quotient P_3 is isomorphic to the non-abelian group of order 27 and exponent 3, K say; note that $\tau(K) = (2, 2, 0)$. This group K has eleven immediate descendants: four of order 81 and seven others. Among the four of order 81, only one group, Q say, has immediate descendants. The descendants of Q all have coclass 1. As observed in Section 3.3, almost all 3-groups of coclass 1 have rank 2. Hence only finitely many descendants of Q are contained in $\mathcal{G}(3, (3, 2, 0))$. It remains to consider the seven other descendants of K. They all have rank 3, width 2, and obliquity 0, but only two of them have immediate descendants with these properties. These two groups are isomorphic to G_4 and H_4, respectively, which are the roots of the τ-trees of $\mathcal{G}(3, (3, 2, 0))$. □

5.1. A conjectural description of the τ-trees.

First we note that G_4 and H_4 are non-isomorphic quotients of G and H which are both contained in $\mathcal{G}(p, (3, 2, 0))$, see [**13**], Section 12.2. Hence the τ-trees $\mathcal{T}_G := \mathcal{T}(G, (3, 2, 0))$ and $\mathcal{T}_H := \mathcal{T}(H, (3, 2, 0))$ are subtrees of $\mathcal{T}(G_4)$ and $\mathcal{T}(H_4)$, respectively. Each tree $\mathcal{T}_G$ or $\mathcal{T}_H$ consists of its unique infinite path and its branches. To describe the two infinite trees, we describe their branches only. Let $B_j(G)$ and $B_j(H)$ denote the branches of $\mathcal{T}_G$ and $\mathcal{T}_H$ with roots G_j and H_j, respectively.

Figures 1–4 exhibit our conjectural descriptions for all of these branches. Their shapes depend on the underlying group G or H, the class of the root of the branch and the underlying prime p. We use a compact notation to visualize trees: if a vertex v has a number n attached to it (written on the right), then there are n vertices $v = v_1, v_2, \ldots, v_n$ in the tree under consideration and all these v_i have the same ancestor and they all have isomorphic descendant trees.

Thus our experiments suggest the following conjecture.

Conjecture 5.2. For every $p > 2$, the graph $\mathcal{G}(p, (3, 2, 0))$ is virtually periodic. More precisely, both of its τ-trees are virtually periodic with period 2 and defect 1.

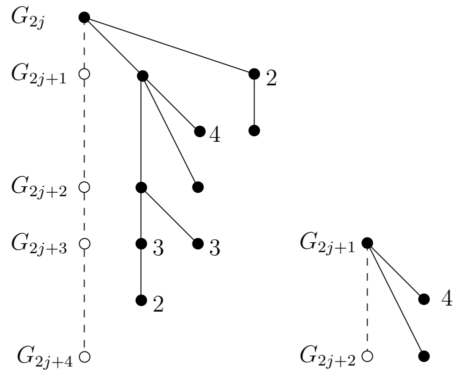


FIGURE 1. The branches $\mathcal{B}_{2j}(G)$ and $\mathcal{B}_{2j+1}(G)$ for $j \geq 2$ and $p = 3$

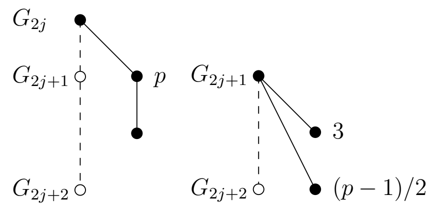


FIGURE 2. The branches $\mathcal{B}_{2j}(G)$ and $\mathcal{B}_{2j+1}(G)$ for $j \geq 2$ and $p \geq 5$

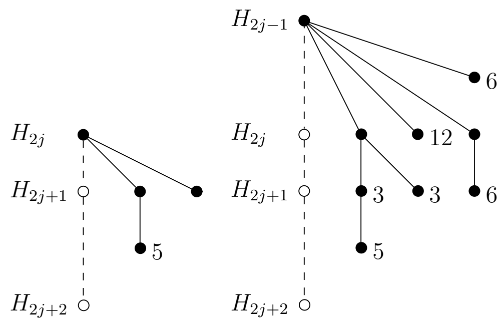


FIGURE 3. The branches $\mathcal{B}_{2j}(H)$ for $j \geq 2$ and $\mathcal{B}_{2j-1}(H)$ for $j \geq 3$ and $p = 3$

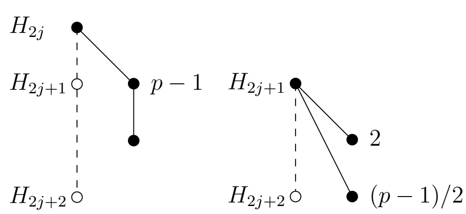


FIGURE 4. The branches $\mathcal{B}_{2j}(H)$ and $\mathcal{B}_{2j+1}(H)$ for $j \geq 2$ and $p \geq 5$

We note that the results of a first investigation of $\mathcal{G}(p, (3, 2, 0))$ for $p \geq 5$ are described in [**13**], Section 12.2, based on results from [**17**]. The results obtained there imply that if $p \geq 5$, then the branches of the τ-trees of $\mathcal{G}(p,(3,2,0))$ have depth at most 2.

5.2. Details on our computations. For given p and i, the quotients G_i and H_i can be constructed readily from their definitions, see also [**10**]. The lower central factors of G and H are known from [**13**], pp. 273–274. In particular, they are always elementary abelian and thus the quotients G_i and H_i are stable for all i. Hence our methods of Section 4 apply.

We determined the first branches of the τ-trees $\mathcal{T}_G$ and $\mathcal{T}_H$ for all primes $p \leq 13$ and the first branches of the full descendant trees $\mathcal{T}(G_4)$ and $\mathcal{T}(H_4)$ for all primes $p \leq 41$. The numbers of branches that we determined for $p \leq 13$ are given in Table 1.

p	$\mathcal{T}_G$	$\mathcal{T}(G_4)$	$\mathcal{T}_H$	$\mathcal{T}(H_4)$
3	11	26	7	6
5	8	18	5	12
7	6	14	10	16
11	5	8	5	10
13	5	6	5	9

Table 1. Numbers of computed branches

The computation of the branches of the full descendant trees is significantly less time-consuming than the corresponding computation for τ-trees, since the determination of the rank and the obliquity for a finite p-group is highly time- and space-consuming. Consequently, we constructed significantly more branches of the full descendants trees than of the corresponding τ-trees. We note that the branches of the descendants trees and the corresponding τ-trees coincided in all cases where we computed both (except for the first branch $\mathcal{B}_4(H)$ in the case $p = 3$).

Our computations were performed on a PC with two 2Ghz processors and 3GB of RAM running under Linux. As an illustration, Table 2 lists approximate runtimes (in seconds) for the construction of the branches $\mathcal{B}_i(G)$ of the τ-tree $\mathcal{T}_G$ and for the corresponding branches $\mathcal{T}(G_i) \setminus \mathcal{T}(G_{i+1})$ of the full descendant trees for $p = 3$.

i	4	5	6	7	8	9	10	11	12	13	14
$\mathcal{B}_i(G)$	59	11	182	32	643	132	2,124	392	6,032	1,149	15,137
$\mathcal{T}(G_i) \setminus \mathcal{T}(G_{i+1})$	11	3	15	3	22	4	31	6	45	9	64

Table 2. Approximate runtimes (in seconds) for G and $p = 3$

6. The trees associated with $s\ell_n(K)$

Let $\mathrm{G}_n(K)$ be the Sylow pro-p-subgroup of $\mathrm{Aut}(L)$, where L is a simple Lie algebra of the form $L = s\ell_n(K)$ for a field extension K of $\mathbb{Q}_p$ with finite degree m. Then $\mathrm{G}_n(K)$ has finite rank, width, and obliquity.

It would be highly interesting to investigate the τ-trees associated with these groups $\mathrm{G}_n(K)$ with a view towards checking whether these trees are virtually periodic. In the smallest case $(n, m) = (2, 1)$ this has been done in Section 5 and a complete conjectural outline of the corresponding trees has been obtained. In all larger cases of (n, m) this is less straightforward with our current computational methods and theoretical knowledge. In this section we consider the groups of the form $\mathrm{G}_n(K)$ for some larger values of (n, m) and investigate their descendant trees as a first approximation of their τ-trees.

The groups $\mathrm{G}_n(K)$ are investigated in [**10**] for the cases with $(n^2 - 1)m \leq 14$; that is, for the cases $(n, m) = (2, 1)$, $(2, 2)$, $(2, 3)$, $(2, 4)$, and $(3, 1)$. In each of these cases there may be several groups depending on the number of different fields K. The different fields K can be retrieved from the database [**9**] for small primes. The following table provides some summary information: it considers the different types of K, the number of corresponding groups $\mathrm{G}_n(K)$ and their parameters $\tau(\mathrm{G}_n(K))$ as far as they are available in [**10**]. If a $\star$ is listed, then the number of fields (and thus the number of groups) depends on the prime.

(n,m)	type of K	$p = 3$		$p \geq 5$	
		# grps	params	# grps	params
$(3,1)$	$K = \mathbb{Q}_p$	1	$(?,2,5)$	1	$(?,3,?)$
$(2,2)$	K totally ramified	2	$(?,2,0)$	2	$(?,2,?)$
$(2,2)$	K unramified	1	$(?,4,0)$	1	$(?,4,?)$
$(2,3)$	K totally ramified	9	$(?,2,0)$, $(?,3,3)$	$\star$	$(?,2,?)$
$(2,3)$	K unramified	1	$(?,3,4)$	1	$(?,6,?)$
$(2,4)$	K totally ramified	2	$(?,2,0)$	$\star$	$(?,2,?)$
$(2,4)$	K mixed ramified	2	$(?,4,0)$	$\star$	$(?,4,?)$
$(2,4)$	K unramified	1	$(?,8,?)$	1	$(?,8,?)$

Among the infinite pro-p-groups of finite rank, width, and obliquity, the groups $\mathrm{G}_n(K)$ are reasonably well understood. The algorithms of [**10**] can be used to determine lower central series quotients of these groups and these can then be used as a basis for further computations. Pc presentations of finite quotients of the groups can be obtained from the Fwtree package [**6**].

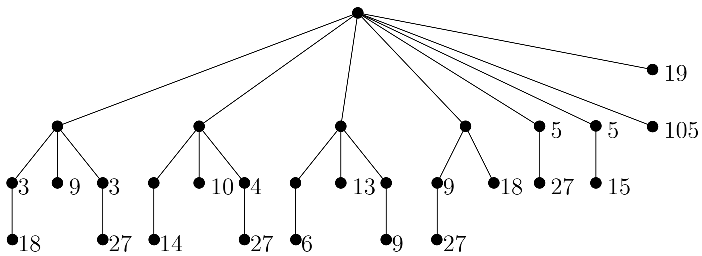
From [**10**] we further obtain that all the groups considered here have stable lower central series quotients so that our algorithms of Section 4 apply and we can compute finite parts of their descendant trees.

6.1. The case $(n, m) = (2, 2)$ and $p = 3$. Consider the groups $\mathrm{G}_2(K)$ with K totally ramified of degree 2 over $\mathbb{Q}_3$. There exist two fields K in this case: $\mathbb{Q}_3(\sqrt{\pm 3})$. Denote the corresponding groups by $G = \mathrm{G}_2(\mathbb{Q}_3(\sqrt{-3}))$ and $H = \mathrm{G}_2(\mathbb{Q}_3(\sqrt{3}))$. Our computational evidence suggests the following conjecture.

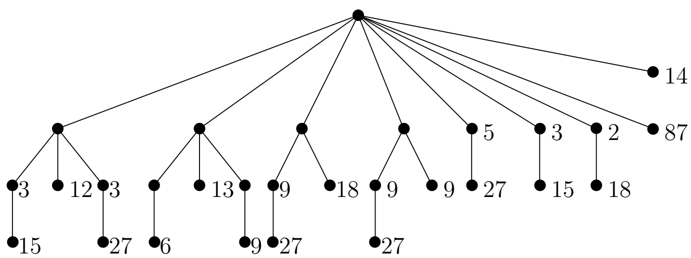
Conjecture 6.1. The descendant trees $\mathcal{T}(G_7)$ and $\mathcal{T}(H_7)$ are both virtually periodic with period 4 and defect 1.

In the following, we display the branches with roots $G_7, \ldots, G_{10}$ and with roots $H_7, \ldots, H_{10}$ using the same notation as in Section 5.1. First, the branches with roots G_7 and H_7 and with roots G_9 and H_9 are isomorphic and are given in the following figures.

G_7, H_7

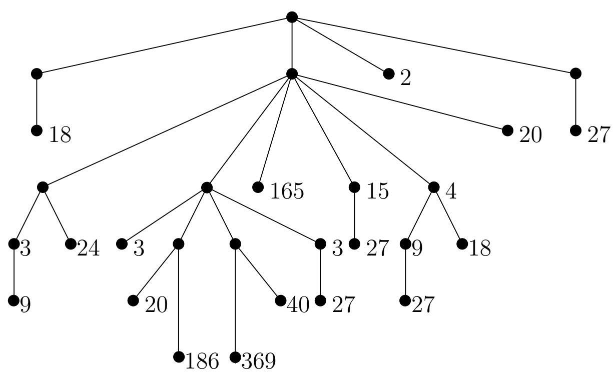


G_9, H_9

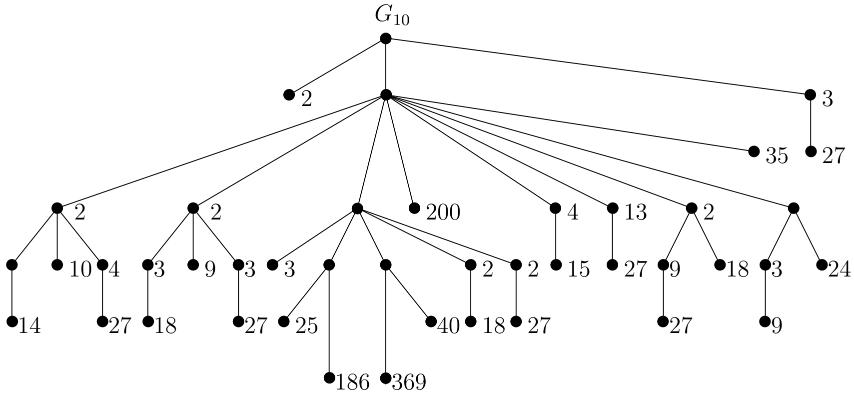


Next, we exhibit the branches with roots G_8 and G_{10}.

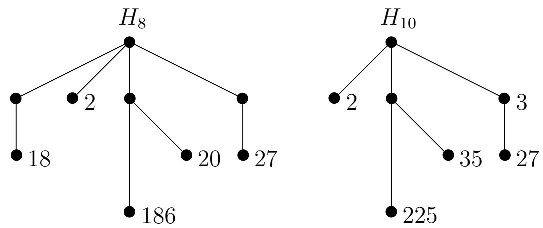
G_8



G_{10}



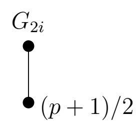
Finally, the branches with roots H_8 and H_{10} are shown in the following figures.



For both choices of the field extension $\mathbb{Q}(\sqrt{\pm 3})$, we have been able to verify five occurrences of the conjectured periodic pattern (consisting of four branches each).

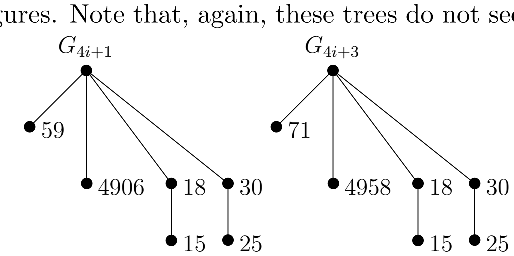
6.2. The case $(n, m) = (2, 2)$ **and** $p > 3$**.** Let $p \geq 5$ and K be a totally ramified extension of degree 2 over $\mathbb{Q}_p$. There are two such fields K to consider and, correspondingly, two groups $\mathrm{G}_2(K)$ exist. Our experimental evidence suggests that there are periodic structures in the descendant trees of the lower central series quotients of these two groups with period a multiple of 4 and defects depending on K.

Let G be either of the two groups $\mathrm{G}_2(K)$. We conjecture that for sufficiently large i the first branch of the descendant tree of G_{2i} is as displayed in the following figure, independent of the field K.



For j odd, we could only properly investigate the branches with root G_j for $p = 5$, and even in this case we have only computed "shaved" versions of these branches. Thus, we have investigated the groups in such a branch of order at most $|G_j|p^3$.

Our conjectural description of the resulting trees in the case $p = 5$ is exhibited in the following figures. Note that, again, these trees do not seem to depend on K.



For $p = 5$, we have been able to verify three full and an incomplete fourth occurrence of the above pattern of four "shaved" branches.

6.3. Other cases. We have also experimented with other cases of (n, m), K, and p. The case $p = 3$ proved to be the most accessible one; we include a brief summary of our experiments with this case. Throughout, let $G = \mathrm{G}_n(K)$ and denote by $\mathcal{T}^*(G_j)$ the subtree of $\mathcal{T}(G_j)$ consisting of the groups H in $\mathcal{T}(G_j)$ satisfying the condition that $\gamma_i(H)/\gamma_{i+1}(H) \cong \gamma_i(G)/\gamma_{i+1}(G)$ holds whenever $\gamma_i(H) \neq 1$.

For $(n, m) = (3, 1)$, there is strong evidence that $\mathcal{T}^*(G_{16})$ is virtually periodic with period 6 and defect 1.

For $(n, m) = (2, 3)$, as in the last subsection, we investigated "shaved" versions of $\mathcal{T}^*(G_j)$ consisting only of those groups H in $\mathcal{T}^*(G_j)$ such that $|H| \leq |G_i|p^d$ for some fixed d. For the six ramified non-Galois extensions of degree 3 over $\mathbb{Q}_3$ we obtained that these "shaved" branches seem to be virtually periodic with period 2; the number d we used varied subject to $3 \leq d \leq 6$.

Similarly, for $(n, m) = (2, 4)$, we also investigated "shaved" branches of $\mathcal{T}^*(G_j)$ only. For the two totally ramified extensions of degree 4 of $\mathbb{Q}_3$ we obtained that the resulting "shaved" branches seem to be virtually periodic with period 4.

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