Computing with nilpotent linear groups

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In this talk, we considered irreducibility testing of nilpotent linear groups over number fields. Let K be a number field and $V \neq 0$ be a finite-dimensional vector space over K. Let $G \leqslant \mathrm{GL}(V)$ be finitely generated and nilpotent. We can assume that G is non-abelian and completely reducible. Our strategy for irreducibility testing of G is built around the following tasks.

(i) Find a proper $K[G]$-submodule of V.

(ii) Find a subspace $U < V$ such that G acts irreducibly on V if and only if $\mathrm{Stab}_G(U)$ acts irreducibly on U, and $\{Ug : g \in G\}$ is a G-system of imprimitivity.

(iii) Find a homogeneous maximal abelian normal subgroup of G.

Using congruence homomorphism techniques and Clifford theory, we can always perform one of these tasks. In case (2), we replace G by the induced linear group acting on U and start again. In case (3), we find that the enveloping algebra of G is in an explicit way a crossed product. We can then decide irreducibility of G directly using computational Galois cohomology [3]; this step, however, is usually non-constructive. Hence, we obtain a "partially constructive" algorithm for irreducibility testing of nilpotent linear groups over K [6, §5.5].

In the case of a nilpotent group $G \leqslant \mathrm{GL}(V)$ which is finite instead of merely finitely generated, we can do much better. In this case, we employ the following variation (based on ideas from [2]) of the above strategy. If G has a non-cyclic abelian normal subgroup, then we can perform task (1) or (2). If, on the other hand, all the abelian normal subgroups of G are cyclic, then the structure of G is sufficiently restricted to allow us to constructively test irreducibility and primitivity of G directly. Consequently, we obtain fully constructive algorithms for both irreducibility and primitivity testing of finite nilpotent linear groups over K [4, 5]. Implementations are included in Magma.

References

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