# Stability results for local zeta functions of groups, algebras, and modules

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Various types of local zeta functions studied in asymptotic group theory admit two natural operations: (1) change the prime and (2) perform local base extensions. Often, the effects of both of the preceding operations can be expressed simultaneously in terms of a single formula, a statement made precise using what we call local maps of Denef type. We show that assuming the existence of such formulae, the behaviour of local zeta functions under variation of the prime in a set of density 1 in fact completely determines these functions for almost all primes and, moreover, it also determines their behaviour under local base extensions. We discuss applications to topological zeta functions, functional equations, and questions of uniformity.

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## 1 Introduction

For a finitely generated nilpotent group $G$ and a prime $p$, let $\zeta^{\widetilde{\mathrm{irr}}}_{G,p}(s)$ be the Dirichlet series enumerating continuous irreducible finite-dimensional complex representations of the pro-$p$ completion $\hat{G}_p$ of $G$, counted up to equivalence and tensoring with continuous 1-dimensional representations. In this article, we prove statements of the following form.

**Theorem.** *Let $G$ and $H$ be finitely generated nilpotent groups such that $\zeta^{\widetilde{\mathrm{irr}}}_{G,p}(s) = \zeta^{\widetilde{\mathrm{irr}}}_{H,p}(s)$ for all primes $p$ in a set of density 1. Then $\zeta^{\widetilde{\mathrm{irr}}}_{G,p}(s) = \zeta^{\widetilde{\mathrm{irr}}}_{H,p}(s)$ for almost all primes $p$.*

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Much more can be said if we also take into account local base extensions. As explained in [19], there exists a unipotent group scheme $\mathsf{G}$ over $\mathbf{Z}$ such that $\hat{G}_p = \mathsf{G}(\mathbf{Z}_p)$ for almost all $p$, where $\mathbf{Z}_p$ denotes the $p$-adic integers; let $\mathsf{H}$ be a unipotent group scheme over $\mathbf{Z}$ associated with $H$ in the same way. The preceding theorem can be sharpened as follows.

**Theorem.** *Suppose that* $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathbf{Z}_p)}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{\mathsf{H}(\mathbf{Z}_p)}}(s)$ *for all primes $p$ in a set of density* 1. *Then* $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathfrak{O}_K)}}(s) = \widetilde{\zeta^{\mathrm{irr}}_{\mathsf{H}(\mathfrak{O}_K)}}(s)$ *for almost all primes $p$ and all finite extensions $K$ of the field $\mathbf{Q}_p$ of $p$-adic numbers, where $\mathfrak{O}_K$ denotes the valuation ring of $K$.*

Explicit formulae in the spirit of Denef's work (see [2]) have been obtained for zeta functions such as $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathfrak{O}_K)}}(s)$ (see [19]). These formulae are well-behaved under variation of the prime $p$ and under local base extensions $K/\mathbf{Q}_p$. More precisely, the family of all such functions $\widetilde{\zeta^{\mathrm{irr}}_{\mathsf{G}(\mathfrak{O}_K)}}(s)$ gives rise to what we call a local map of Denef type. Informally, a local map of Denef type $\mathsf{Z}$ uses an algebro-geometric template to assign to each prime $p$ (with possibly finitely many exceptions) and number $f \geqslant 1$ a rational function $\mathsf{Z}(p, f)$ corresponding to the unramified extension of $\mathbf{Q}_p$ of degree $f$; see §2.1 for a precise definition. In the cases of interest to us, $\mathsf{Z}(p, f)$ will be a local zeta function derived from some global object (e.g. a group scheme $\mathsf{G}$ as above). Our main result, Theorem 2.3, shows that up to a suitable notion of equivalence, $\mathsf{Z}$ is determined by the rational functions $\mathsf{Z}(p, 1)$ as $p$ ranges over the elements of a set of primes of density 1. We prove this by interpreting $\mathsf{Z}$ in terms of Galois representations and by invoking Chebotarev's density theorem. Applications of these techniques are given in [17] which provides the main inspiration for the present article. In particular, Theorem 2.3 draws upon the following.

**Theorem** ([17, Thm 1.3])**.** *Let $V$ and $W$ be schemes of finite type over $\mathbf{Z}$. Suppose that $\#V(\mathbf{F}_p) = \#W(\mathbf{F}_p)$ for all $p$ in a set of primes of density* 1. *Then $\#V(\mathbf{F}_{p^f}) = \#W(\mathbf{F}_{p^f})$ for almost all primes $p$ and all $f \in \mathbf{N}$.*

After proving Theorem 2.3, in §3, we consider consequences to topological zeta functions. In particular, we give a rigorous justification for the process of deriving topological zeta functions from uniform $p$-adic ones. In §4, we show that local functional equations under "inversion of $p$" are independent of the formulae used to define them.

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### *Notation*

We write $\mathbf{N} = \{1, 2, \dots\}$ and let "$\subset$" indicate not necessarily proper inclusion. Throughout, $k$ is a number field with ring of integers $\mathfrak{o}$. Let $\mathcal{V}_k$ be the set of non-Archimedean places of $k$. Given $v \in \mathcal{V}_k$, let $k_v$ be the $v$-adic completion of $k$ and $\mathfrak{K}_v$ be its residue field. Let $q_v$ and $p_v$ denote the cardinality and characteristic of $\mathfrak{K}_v$, respectively. Let $\bar{k}_v$ be an algebraic closure of $k_v$. For $f \geqslant 1$, let $k_v^{(f)} \subset \bar{k}_v$ be the unramified extension of degree $f$ of $k_v$; we identify the residue field $\mathfrak{K}_v^{(f)}$ of $k_v^{(f)}$ with the extension of degree $f$ of $\mathfrak{K}_v$

within the residue field of $\bar{k}_v$. Let $\mathfrak{O}_K$ denote the valuation ring of a non-Archimedean local field $K$, let $\mathfrak{P}_K$ denote the maximal ideal of $\mathfrak{O}_K$, and write $q_K = \#(\mathfrak{O}_K/\mathfrak{P}_K)$. By the density of a set of places or primes, we mean the Dirichlet density.

# 2 Stability under base extension for local maps of Denef type

This section contains the technical main result of this article: Theorem 2.3.

## 2.1 Local maps of Denef type

The following terminology is closely related to [13, §5.2]. Fix $m \in \mathbf{N}$. A **$k$-local map** is a map $\mathsf{Z}\colon (\mathcal{V}_k \setminus S_{\mathsf{Z}}) \times \mathbf{N} \to \mathbf{Q}(Y_1, \dots, Y_m)$, where $S_{\mathsf{Z}} \subset \mathcal{V}_k$ is finite. Local maps provide a convenient formalism for studying families of local zeta functions: for $v \in \mathcal{V}_k \setminus S_{\mathsf{Z}}$ and $f \in \mathbf{N}$, let $\hat{\mathsf{Z}}(v, f)$ denote the meromorphic function $\mathsf{Z}(v, f)(q_v^{-fs_1}, \dots, q_v^{-fs_m})$ in complex variables $s_1, \dots, s_m$. Let $K$ be a non-Archimedean local field endowed with an embedding $k \subset K$. We may regard $K$ as an extension of $k_v$ for a unique $v \in \mathcal{V}_k$. Let $f$ be the inertia degree of $K/k_v$. Given a $k$-local map $\mathsf{Z}$, if $v \notin S_{\mathsf{Z}}$, write $\mathsf{Z}_K := \mathsf{Z}(v, f)$ and $\hat{\mathsf{Z}}_K(s_1, \dots, s_m) := \hat{\mathsf{Z}}(v, f)$.

We say that two $k$-local maps $\mathsf{Z}$ and $\mathsf{Z}'$ are **equivalent** if they coincide on $(\mathcal{V}_k \setminus S) \times \mathbf{N}$ for some finite set $S \supset S_{\mathsf{Z}} \cup S_{\mathsf{Z}'}$. Let $V$ be a separated $\mathfrak{o}$-scheme of finite type and let $W \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ be regular at $(q, Y_1, \dots, Y_m)$ for each integer $q > 1$. Define

$$[V \cdot W]\colon \mathcal{V}_k \times \mathbf{N} \to \mathbf{Q}(Y_1, \dots, Y_m), \quad (v, f) \mapsto \#V(\mathfrak{K}_v^{(f)}) \cdot W(q_v^f, Y_1, \dots, Y_m).$$

We say that a $k$-local map is of **Denef type** if it is equivalent to a finite (pointwise) sum of maps of the form $[V \cdot W]$. (For a motivation of our terminology, see [2, §3].)

## 2.2 Main examples of local maps

We discuss the primary examples of $k$-local maps of Denef type of interest to us. These local maps will be constructed from an $\mathfrak{o}$-form of a $k$-object and only be defined up to equivalence. In the univariate case $m = 1$, we simply write $Y = Y_1$ and $s = s_1$.

**Example 2.1** (Subalgebra and submodule zeta functions [6])**.** The following goes back to [9, 18], see [21] for a survey; we use the formalism from [13, §2.1].

(i) Let $R$ be a compact discrete valuation ring and let $\mathsf{A}$ be a possibly non-associative $R$-algebra whose underlying $R$-module is finitely generated. For $n \in \mathbf{N}$, let $a_n^{\leqslant}(\mathsf{A})$ denote the number of subalgebras $\mathsf{U}$ of $\mathsf{A}$ such that the $R$-module $\mathsf{A}/\mathsf{U}$ has cardinality $n$. The **subalgebra zeta function** of $\mathsf{A}$ is $\zeta_{\mathsf{A}}^{\leqslant}(s) = \sum_{n=1}^{\infty} a_n^{\leqslant}(\mathsf{A}) n^{-s}$.

   Let $\mathcal{A}$ be a possibly non-associative finite-dimensional $k$-algebra. Choose an $\mathfrak{o}$-form $\mathsf{A}$ of $\mathcal{A}$. By [6, Thm 1.4] (cf. [13, Thm 5.16]), there is a $k$-local map $\mathsf{Z}^{\mathcal{A}}\colon (\mathcal{V}_k \setminus S) \times \mathbf{N} \to \mathbf{Q}(Y)$ of Denef type with $\hat{\mathsf{Z}}_K^{\mathcal{A}}(s) = \zeta_{\mathsf{A} \otimes_{\mathfrak{o}} \mathfrak{O}_K}^{\leqslant}(s)$ for all non-Archimedean local fields $K$ which extend $k$. The equivalence class of $\mathsf{Z}^{\mathcal{A}}$ only depends on $\mathcal{A}$.

(ii) Let $R$ be a compact discrete valuation ring, $\mathsf{M}$ be a finitely generated $R$-module, and let $\mathsf{E}$ be an associative unital $R$-subalgebra of $\mathrm{End}_R(\mathsf{M})$. For $n \in \mathbf{N}$, let $b_n(\mathsf{E} \curvearrowright \mathsf{M})$ denote the number of $\mathsf{E}$-submodules $\mathsf{U} \subset \mathsf{M}$ such that the $R$-module $\mathsf{M}/\mathsf{U}$ has cardinality $n$. The **submodule zeta function** of $\mathsf{E}$ acting on $\mathsf{M}$ is $\zeta_{\mathsf{E}\curvearrowright\mathsf{M}}(s) = \sum_{n=1}^{\infty} b_n(\mathsf{E} \curvearrowright \mathsf{M}) n^{-s}$. Let $M$ be a finite-dimensional vector space over $k$, and let $\mathcal{E}$ be an associative subalgebra of $\mathrm{End}_k(M)$. Choose an $\mathfrak{o}$-form $\mathsf{M}$ of $M$ and an $\mathfrak{o}$-form $\mathsf{E} \subset \mathrm{End}_{\mathfrak{o}}(\mathsf{M})$ of $\mathcal{E}$. Similarly to (i), we obtain a $k$-local map $\mathsf{Z}^{\mathcal{E}\curvearrowright M} \colon (\mathcal{V}_k \setminus S) \times \mathbf{N} \to \mathbf{Q}(Y)$ of Denef type with $\hat{\mathsf{Z}}_K^{\mathcal{E}\curvearrowright M}(s) = \zeta_{(\mathsf{E}\otimes_{\mathfrak{o}}\mathfrak{O}_K)\curvearrowright(\mathsf{M}\otimes_{\mathfrak{o}}\mathfrak{O}_K)}(s)$ for all non-Archimedean local fields $K$ which extend $k$.

**Example 2.2** (Representation zeta functions)**.** For details on the following, we refer to [11] and [21, §4]. Let $G$ be a topological group. For $n \in \mathbf{N}$, let $r_n(G) \in \mathbf{N} \cup \{0, \infty\}$ denote the number of equivalence classes of continuous irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$. Supposing that $r_n(G) < \infty$ for all $n$, the **representation zeta function** of $G$ is $\zeta_G^{\mathrm{irr}}(s) = \sum_{n=1}^{\infty} r_n(G) n^{-s}$. Two continuous complex representations $\varrho$ and $\sigma$ of $G$ are **twist-equivalent** if $\varrho$ is equivalent to $\sigma \otimes_{\mathbf{C}} \alpha$ for a continuous 1-dimensional complex representation $\alpha$ of $G$. For $n \in \mathbf{N}$, let $\tilde{r}_n(G)$ denote the number of twist-equivalence classes of continuous irreducible representations $G \to \mathrm{GL}_n(\mathbf{C})$. Supposing that $\tilde{r}_n(G) < \infty$ for all $n$, the **twist-representation zeta function** of $G$ is $\zeta_G^{\widetilde{\mathrm{irr}}}(s) = \sum_{n=1}^{\infty} \tilde{r}_n(G) n^{-s}$.

(i) (See [19].) Let $\mathbf{G}$ be a unipotent algebraic group over $k$. Choose an affine group scheme $\mathsf{G}$ of finite type over $\mathfrak{o}$ such that $\mathsf{G} \otimes_{\mathfrak{o}} k$ and $\mathbf{G}$ are isomorphic over $k$. There exists a finite set $S \subset \mathcal{V}_k$ such that $\mathsf{G}(\mathfrak{O}_K)$ is a finitely generated nilpotent pro-$p_v$ group for $v \in \mathcal{V}_k \setminus S$ and all finite extensions $K/k_v$. By [19, Pf of Thm A], after enlarging $S$, we obtain a $k$-local map $\mathsf{Z}^{\mathbf{G},\widetilde{\mathrm{irr}}} \colon (\mathcal{V}_k \setminus S) \times \mathbf{N} \to \mathbf{Q}(Y)$ of Denef type with $\hat{\mathsf{Z}}_K^{\mathbf{G},\widetilde{\mathrm{irr}}}(s) = \zeta_{\mathsf{G}(\mathfrak{O}_K)}^{\widetilde{\mathrm{irr}}}(s)$; the equivalence class of $\mathsf{Z}^{\mathbf{G},\widetilde{\mathrm{irr}}}$ only depends on $\mathbf{G}$.

(ii) (See [1].) Let $\mathfrak{g}$ be a finite-dimensional perfect Lie $k$-algebra. Choose an $\mathfrak{o}$-form $\mathsf{g}$ of $\mathfrak{g}$. As explained in [1, §2.1], there exists a finite set $S \subset \mathcal{V}_k$ such that for each $v \in \mathcal{V}_k \setminus S$ and $f \in \mathbf{N}$, the set $\mathsf{G}^1(\mathfrak{o}_v^{(f)}) := \mathfrak{p}_v^{(f)}(\mathsf{g} \otimes_{\mathfrak{o}} \mathfrak{o}_v^{(f)})$ can be naturally endowed with the structure of a FAb $p_v$-adic analytic pro-$p_v$ group; here, $\mathfrak{o}_v^{(f)}$ is the valuation ring of $k_v^{(f)}$ and $\mathfrak{p}_v^{(f)}$ the maximal ideal of $\mathfrak{o}_v^{(f)}$. By [1, §§3–4], after enlarging $S$, we obtain a map of Denef type $\mathsf{Z}^{\mathfrak{g},\mathrm{irr}} \colon (\mathcal{V}_k \setminus S) \times \mathbf{N} \to \mathbf{Q}(Y)$ with $\hat{\mathsf{Z}}^{\mathfrak{g},\mathrm{irr}}_{k_v^{(f)}}(s) = \zeta^{\mathrm{irr}}_{\mathsf{G}^1(\mathfrak{o}_v^{(f)})}(s)$, the equivalence class of which only depends on $\mathfrak{g}$.

We note that there are natural algebraic counting problems that give rise to local zeta functions but not to local maps. For example, it follows from work of Segal [16] that the zeta function enumerating ideals of finite additive index of $\mathbf{Z}_p[T]$ is not rational in $p^{-s}$.

## 2.3 Main results

The following result, which we will prove in §2.4, constitutes the technical heart of this article. As before, let $k$ be a number field with ring of integers $\mathfrak{o}$.

**Theorem 2.3.** *Let $V_1, \dots, V_r$ be separated $\mathfrak{o}$-schemes of finite type and let $W_1, \dots, W_r \in \mathbf{Q}(X, Y_1, \dots, Y_m)$. Suppose that $(q, Y_1, \dots, Y_m)$ is a regular point of each $W_i$ for each integer $q > 1$. Let $P \subset \mathcal{V}_k$ be a set of places of density 1 and suppose that for all $v \in P$, we have $\sum_{i=1}^{r} \#V_i(\mathfrak{K}_v) \cdot W_i(q_v, Y_1, \dots, Y_m) = 0$. Then there exists a finite set $S \subset \mathcal{V}_k$ such that for all $v \in \mathcal{V}_k \setminus S$ and all $f \in \mathbf{N}$, we have $\sum_{i=1}^{r} \#V_i(\mathfrak{K}_v^{(f)}) \cdot W_i(q_v^f, Y_1, \dots, Y_m) = 0$.*

We now discuss consequences of Theorem 2.3 for local maps of Denef type. The following implies the first two theorems stated in the introduction.

**Corollary 2.4.** *Let $\mathsf{Z}, \mathsf{Z}'$ be $k$-local maps of Denef type. Let $P \subset \mathcal{V}_k \setminus (S_{\mathsf{Z}} \cup S_{\mathsf{Z}'})$ have density 1 and let $\mathsf{Z}(v, 1) = \mathsf{Z}'(v, 1)$ for all $v \in P$. Then $\mathsf{Z}$ and $\mathsf{Z}'$ are equivalent. That is, there exists a finite set $S \supset S_{\mathsf{Z}} \cup S_{\mathsf{Z}'}$ with $\mathsf{Z}(v, f) = \mathsf{Z}'(v, f)$ for all $v \in \mathcal{V}_k \setminus S$ and $f \in \mathbf{N}$.*

*Proof.* Apply Theorem 2.3 to the difference $\mathsf{Z} - \mathsf{Z}'$. ♦

**Example 2.5.** Let $\mathsf{H}(R) = \begin{bmatrix} 1 & R & R \\ 0 & 1 & R \\ 0 & 0 & 1 \end{bmatrix}$ be the Heisenberg group scheme. It follows from work of Nunley and Magid [12] that $\zeta^{\widetilde{\mathrm{irr}}}_{\mathsf{H}(\mathbf{Z}_p)}(s) = \frac{1-p^{-s}}{1-p^{1-s}}$ for all $p$. Using Corollary 2.4 and Example 2.2, we deduce that $\zeta_{\mathsf{H}(\mathfrak{O}_K)}(s) = \frac{1-q_K^{-s}}{1-q_K^{1-s}}$ for almost all $p$ and all finite extensions $K/\mathbf{Q}_p$. Stasinski and Voll [19, Thm B] proved that no primes need to be excluded here.

In analogy with [8, §1.2.4], we say that a $k$-local map $\mathsf{Z}$ is **uniform** if it is equivalent to $(v, f) \mapsto W(q_v^f, Y_1, \dots, Y_m)$ for some $W \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ which is regular at each point $(q, Y_1, \dots, Y_m)$ for each integer $q > 1$. We then say that $W$ **uniformly represents** $\mathsf{Z}$.

**Corollary 2.6.** *Let $\mathsf{Z}$ be a $k$-local map of Denef type. Let $P \subset \mathcal{V}_k \setminus S_{\mathsf{Z}}$ have density 1 and let $W \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ be regular at $(q, Y_1, \dots, Y_m)$ for all integers $q > 1$. If $\mathsf{Z}(v, 1) = W(q_v, Y_1, \dots, Y_m)$ for all $v \in P$, then $W$ uniformly represents $\mathsf{Z}$.*

*Proof.* Apply Corollary 2.4 with $\mathsf{Z}'(v, f) = W(q_v^f, Y_1, \dots, Y_m)$. ♦

This, in particular, applies to a large number of examples of uniform $p$-adic subalgebra and ideal zeta functions computed by Woodward [8]. Specifically, [8, Ch. 2] contains numerous examples of Lie rings $\mathsf{L}$ such that $\zeta^{\leqslant}_{\mathsf{L} \otimes \mathbf{Z}_p}(s) = W(p, p^{-s})$ for some $W \in \mathbf{Q}(X, Y)$ and all rational primes $p$ (or almost all of them). Corollary 2.6 shows that for almost all $p$ and all finite extensions $K/\mathbf{Q}_p$, we then have $\zeta^{\leqslant}_{\mathsf{L} \otimes \mathfrak{O}_K}(s) = W(q_K, q_K^{-s})$.

## 2.4 Proof of Theorem 2.3

We first recall some facts from [17, Ch. 3–4]. Let $G$ be a profinite group. We let $\mathrm{Cl}(g)$ denote the conjugacy class of $g \in G$ and write $\mathrm{Cl}(G) := \{\mathrm{Cl}(g) : g \in G\}$. We may naturally regard $\mathrm{Cl}(G)$ as a profinite space by endowing it with the quotient topology or, equivalently, by identifying $\mathrm{Cl}(G) = \varprojlim_{N \triangleleft_o G} \mathrm{Cl}(G/N)$, where $N$ ranges over the open normal subgroups of $G$ and each $\mathrm{Cl}(G/N)$ is regarded as a discrete space.

As before, let $k$ be a number field with ring of integers $\mathfrak{o}$. Fix an algebraic closure $\bar{k}$ of $k$. For a finite set $S \subset \mathcal{V}_k$, let $\Gamma_S$ denote the Galois group of the maximal extension of $k$ within $\bar{k}$ which is unramified outside of $S$. For $v \in \mathcal{V}_k \setminus S$, choose $g_v \in \Gamma_S$ in the conjugacy class of geometric Frobenius elements associated with $v$, see [17, §§4.4, 4.8.2]. The following is a consequence of Chebotarev's density theorem.

**Theorem 2.7** (Cf. [17, Thm 6.7])**.** *Let $P \subset \mathcal{V}_k \setminus S$ have density* 1*. Then* $\{\mathrm{Cl}(g_v) : v \in P\}$ *is a dense subset of* $\mathrm{Cl}(\Gamma_S)$*.*

While only natural densities are considered in [17] (see [17, §§1.3, 3.1.3]), the proof of [17, Thm 6.7] implies Theorem 2.7 in the form stated here (i.e. for Dirichlet densities).

Fix a prime $\ell$. Recall that a **virtual $\ell$-adic character** of a profinite group $G$ is a map $\alpha\colon G \to \mathbf{Q}_\ell$ of the form $g \mapsto \sum_{i=1}^{u} c_i \operatorname{trace}(\varrho_i(g))$, where $c_1, \dots, c_u \in \mathbf{Z}$ and each $\varrho_i$ is a continuous homomorphism from $G$ into some $\mathrm{GL}_{n_i}(\mathbf{Q}_\ell)$. Note that such a map $\alpha$ induces a continuous map $\mathrm{Cl}(G) \to \mathbf{Q}_\ell$.

The next result follows from Grothendieck's trace formula [17, Thm 4.2] and the generic cohomological behaviour of reduction modulo non-zero primes of $\mathfrak{o}$ [17, Thm 4.13].

**Theorem 2.8** (See [17, Ch. 4] and cf. [17, §§6.1.1–6.1.2])**.** *Let $V$ be a separated $\mathfrak{o}$-scheme of finite type. Then there exist a finite set $S \subset \mathcal{V}_k$ and a virtual $\ell$-adic character $\alpha$ of $\Gamma_S$ such that $\#V(\mathfrak{K}_v^{(f)}) = \alpha(g_v^f)$ for all $v \in \mathcal{V}_k \setminus S$ and all $f \in \mathbf{N}$.*

Note that since $\#V(\mathfrak{K}_v) \in \mathbf{N} \cup \{0\}$ for all $v \in \mathcal{V}_k$ and $\{\mathrm{Cl}(g_v) : v \in \mathcal{V}_k \setminus S\}$ is dense in $\mathrm{Cl}(\Gamma_S)$, the virtual character $\alpha$ in Theorem 2.8 is $\mathbf{Z}_\ell$-valued.

*Proof of Theorem 2.3.* There exists a non-zero $D \in \mathbf{Z}[X, Y_1, \dots, Y_m]$ such that $DW_i \in \mathbf{Z}[X, Y_1, \dots, Y_m]$ for $i = 1, \dots, r$ and $D(q, Y_1, \dots, Y_m) \neq 0$ for integers $q > 1$. The proof of Theorem 2.3 is thus reduced to the case $W_1, \dots, W_r \in \mathbf{Z}[X, Y_1, \dots, Y_m]$. By considering the coefficients of each monomial $Y_1^{a_1} \cdots Y_m^{a_m}$, we may assume that $W_1, \dots, W_r \in \mathbf{Z}[X]$.

Let $V_0 := \mathbf{A}^1_{\mathfrak{o}}$. By Theorem 2.8, there exists a finite set $S \subset \mathcal{V}_k$ and for $0 \leqslant i \leqslant r$, a continuous virtual $\ell$-adic character $\gamma_i\colon \Gamma_S \to \mathbf{Z}_\ell$ with $\#V_i(\mathfrak{K}_v^{(f)}) = \gamma_i(g_v^f)$ for all $v \in \mathcal{V}_k \setminus S$ and $f \in \mathbf{N}$. By construction, the virtual character $\alpha := \sum_{i=1}^{r} \gamma_i \cdot W_i(\gamma_0)\colon \Gamma_S \to \mathbf{Z}_\ell$ satisfies $\alpha(g_v) = 0$ for $v \in P \setminus S$ whence $\alpha = 0$ by Theorem 2.7. The theorem now follows by evaluating $\alpha$ at the $g_v^f$. ♦

**Lemma 2.9.** *In the setting of Theorem 2.3, suppose that each $V_i \otimes_{\mathfrak{o}} k$ is smooth and proper over $k$. Let $S' \subset \mathcal{V}_k$ such that $v \in \mathcal{V}_k \setminus S'$ if and only if $V_i \otimes \mathfrak{K}_v$ is smooth and proper over $\mathfrak{K}_v$ for each $i = 1, \dots, r$. Then we may take $S = S'$ in Theorem 2.3.*

*Proof.* By the proof of Theorem 2.3 and [17, §4.8.4], for every prime $\ell$, we may take $S = S' \cup \{v \in \mathcal{V}_k : p_v = \ell\}$ in Theorem 2.3. The claim follows since $\ell$ is arbitrary. ♦

## 3 Application: topological zeta functions and $p$-adic formulae

Denef and Loeser introduced topological zeta functions of polynomials using an $\ell$-adic limit "$p \to 1$" of Igusa's $p$-adic zeta functions, see [3]; for an approach using motivic integration, see [4]. Based on the motivic point of view, du Sautoy and Loeser [7, §8] introduced topological subalgebra zeta functions. In this section, we show that $p$-adic formulae determine associated topological zeta functions and we discuss consequences.

We recall the formalism from [13, §5] as in [15, §3.1]. For $e \in \mathbf{Q}[s_1, \dots, s_m]$, the expansion $X^e := \sum_{d=0}^{\infty} \binom{e}{d}(X-1)^d \in \mathbf{Q}[s_1, \dots, s_m][\![X-1]\!]$ yields an embedding $h \mapsto h(X, X^{-s_1}, \dots, X^{-s_m})$ of $\mathbf{Q}(X, Y_1, \dots, Y_m)$ into $\mathbf{Q}(s_1, \dots, s_m)(\!(X-1)\!)$. Let $\mathbf{M}[X, Y_1, \dots, Y_m]$ be the subalgebra of $\mathbf{Q}(X, Y_1, \dots, Y_m)$ consisting of those $W = g/h$ with $W(X, X^{-s_1}, \dots, X^{-s_m}) \in \mathbf{Q}(s_1, \dots, s_m)[\![X-1]\!]$, where $g \in \mathbf{Q}[X^{\pm 1}, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ and $h$ is a finite product of non-zero factors $1 - X^a Y_1^{b_1} \cdots Y_m^{b_m}$ for $a, b_1, \dots, b_m \in \mathbf{Z}$. Given $W \in \mathbf{M}[X, Y_1, \dots, Y_m]$, write $\lfloor W \rceil := W(X, X^{-s_1}, \dots, X^{-s_m}) \bmod (X-1) \in \mathbf{Q}(s_1, \dots, s_m)$. We say that a $k$-local map $\mathsf{Z}$ is **expandable** if it is equivalent to a sum of maps $[V \cdot W]$, where $V$ is a separated $\mathfrak{o}$-scheme of finite type and $W \in \mathbf{M}[X, Y_1, \dots, Y_m]$.

**Example 3.1.**

(i) The local maps $\mathsf{Z}^{\mathbf{G}, \widetilde{\mathrm{irr}}}$ and $\mathsf{Z}^{\mathfrak{g}, \mathrm{irr}}$ from §2.2 are expandable, see [15, §§3.2–3.3].

(ii) If $\mathcal{A}$ and $M$ from Example 2.1 have $k$-dimension $d$, then the local maps $(1-X^{-1})^d \mathsf{Z}^{\mathcal{A}}$ and $(1-X^{-1})^d \mathsf{Z}^{\mathcal{E} \curvearrowright M}$ (pointwise products) are both expandable by [13, Thm 5.16].

If $V$ is a $k$-variety, then any embedding $k \hookrightarrow \mathbf{C}$ allows us to regard $V(\mathbf{C})$ as a $\mathbf{C}$-analytic space. Comparison theorems (see e.g. [10]) show that the topological Euler characteristic $\chi(V(\mathbf{C}))$ does not depend on the embedding. The following formalises insights from [3].

**Theorem 3.2** (Cf. [13, Thm 5.12])**.** *Let $V_1, \dots, V_r$ be separated $\mathfrak{o}$-schemes of finite type, let $W_1, \dots, W_r \in \mathbf{M}[X, Y_1, \dots, Y_m]$, and let $S \subset \mathcal{V}_k$ be finite. Suppose that for all $v \in \mathcal{V}_k \setminus S$ and $f \in \mathbf{N}$, we have $\sum\limits_{i=1}^{r} \#V_i(\mathfrak{K}_v^{(f)}) \cdot W_i(q_v^f, Y_1, \dots, Y_m) = 0$. Then $\sum\limits_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rceil = 0$.*

Let $\mathsf{Z}$ be a $k$-local map equivalent to $[V_1 \cdot W_1] + \dots + [V_r \cdot W_r]$, where $V_1, \dots, V_r$ are separated $\mathfrak{o}$-schemes of finite type and $W_1, \dots, W_r \in \mathbf{M}[X, Y_1, \dots, Y_m]$. By Theorem 3.2, we may unambiguously define the **topological zeta function** $\mathsf{Z}_{\mathrm{top}} \in \mathbf{Q}(s_1, \dots, s_m)$ associated with $\mathsf{Z}$ via $\mathsf{Z}_{\mathrm{top}} := \sum_{i=1}^{r} \chi(V_i(\mathbf{C})) \cdot \lfloor W_i \rceil$. By applying this definition to the local maps in Example 3.1, topological versions of the zeta functions from §2.2 are defined.

**Corollary 3.3.** *Let $\mathsf{Z}$ and $\mathsf{Z}'$ be $k$-local maps of Denef type. Let $P \subset \mathcal{V}_k \setminus (S_{\mathsf{Z}} \cup S_{\mathsf{Z}'})$ have density 1 and suppose that $\mathsf{Z}(v, 1) = \mathsf{Z}'(v, 1)$ for all $v \in P$. If $\mathsf{Z}$ is expandable, then so is $\mathsf{Z}'$ and $\mathsf{Z}_{\mathrm{top}} = \mathsf{Z}'_{\mathrm{top}}$.*

*Proof.* Combine Corollary 2.4 and Theorem 3.2. ♦

For instance, given $\mathbf{Z}$-forms $\mathsf{A}$ and $\mathsf{B}$ of $\mathbf{Q}$-algebras of the same dimension, if $\zeta^{\leqslant}_{\mathsf{A} \otimes \mathbf{Z}_p}(s) = \zeta^{\leqslant}_{\mathsf{B} \otimes \mathbf{Z}_p}(s)$ for almost all $p$, then $\mathsf{A}$ and $\mathsf{B}$ have the same topological subalgebra zeta function.

In the introductions to [13–15], the author informally "read off" topological zeta functions from $p$-adic formulae. The informal nature was due to some of these formulae only being known under variation of $p$ but not under base extension. Corollary 3.3 and the following lemma show that this approach is fully rigorous.

**Lemma 3.4.** *Let $W = g(X, Y_1, \dots, Y_m)/\prod_{i\in I}(1 - X^{a_i} Y_1^{b_{i1}} \cdots Y_m^{b_{im}}) \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ for a Laurent polynomial $g \in \mathbf{Q}[X^{\pm 1}, Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$, a finite set $I$, and integers $a_i, b_{ij}$ with $(a_i, b_{i1}, \dots, b_{im}) \neq 0$ for $i \in I$. Let $\mathsf{Z}$ be an expandable $k$-local map which is uniformly represented by $W$. Then $W \in \mathbf{M}[X, Y_1, \dots, Y_m]$ and thus $\mathsf{Z}_{\mathrm{top}} = \lfloor W \rfloor$.*

*Proof.* By [13, Thm 5.12], there exists a finite union of affine hyperplanes $\mathcal{H} \subset \mathbf{A}^m_{\mathbf{Z}}$ such that for any prime $\ell$ and almost all $v \in \mathcal{V}_k$, there exists $d \in \mathbf{N}$ such that

$$\mathbf{N} \times (\mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z})) \to \mathbf{Q}, \quad (f; \boldsymbol{s}) \mapsto \hat{\mathsf{Z}}_{k_v^{(df)}}(\boldsymbol{s}_1, \dots, \boldsymbol{s}_m)$$

is well-defined and admits a continuous extension $\Phi \colon \mathbf{Z}_\ell \times (\mathbf{Z}_\ell^m \setminus \mathcal{H}(\mathbf{Z}_\ell)) \to \mathbf{Q}_\ell$. We may assume that $a_i \neq \sum_{j=1}^m b_{ij}\boldsymbol{s}_j$ for $i \in I$ and $\boldsymbol{s} \in \mathbf{Z}_\ell^m \setminus \mathcal{H}(\mathbf{Z}_\ell)$. Since $W$ uniformly represents $\mathsf{Z}$, we may choose $(\mathcal{H}, \ell, v, d)$ such that $\Phi(f; \boldsymbol{s}) = W(q_v^{df}, q_v^{-df\boldsymbol{s}_1}, \dots, q_v^{-df\boldsymbol{s}_m})$ for $(f; \boldsymbol{s}) \in \mathbf{N} \times (\mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z}))$. Finally, we may also assume that $\ell \neq 2$ and that $q_v^d \equiv 1 \bmod \ell$.

Let $w$ be the $(X-1)$-adic valuation of $g(X, X^{-s_1}, \dots, X^{-s_m}) \in \mathbf{Q}[s_1, \dots, s_m][\![X-1]\!]$. Define $G(s_1, \dots, s_m; X-1) \in \mathbf{Q}[s_1, \dots, s_m][\![X-1]\!]$ by $g(X, X^{-s_1}, \dots, X^{-s_m}) = (X-1)^w \cdot G(s_1, \dots, s_m; X-1)$ and note that $G(s_1, \dots, s_m; 0) \neq 0$. As in the proof of [13, Lem. 5.6], using the $\ell$-adic binomial series, for $f \in \mathbf{Z}_\ell$ and $\boldsymbol{s} \in \mathbf{Z}_\ell^m$, we have

$$g((q_v^d)^f, (q_v^d)^{-f\boldsymbol{s}_1}, \dots, (q_v^d)^{-f\boldsymbol{s}_m}) = ((q_v^d)^f - 1)^w \cdot G(\boldsymbol{s}; (q_v^d)^f - 1). \tag{3.1}$$

Let $\boldsymbol{s}_\infty \in \mathbf{Z}_\ell^m \setminus \mathcal{H}(\mathbf{Z}_\ell)$ such that $G(s_1, \dots, s_m; 0)$ does not vanish at $\boldsymbol{s}_\infty$. Let $(f_n)_{n\in\mathbf{N}} \subset \mathbf{N}$ with $f_n \to 0$ in $\mathbf{Z}_\ell$ and $(\boldsymbol{s}_n)_{n\in\mathbf{N}} \subset \mathbf{Z}^m \setminus \mathcal{H}(\mathbf{Z})$ with $\boldsymbol{s}_n \to \boldsymbol{s}_\infty$ in $\mathbf{Z}_\ell^m$. Let $x_n := q_v^{df_n}$. Then

$$W(x_n, x_n^{-\boldsymbol{s}_{n1}}, \dots, x_n^{-\boldsymbol{s}_{nm}}) = \Phi(f_n; \boldsymbol{s}_n) \to \Phi(0; \boldsymbol{s}_\infty) \in \mathbf{Q}_\ell \tag{3.2}$$

as $n \to \infty$. Let $e_{in} := a_i - b_{i1}\boldsymbol{s}_{n1} - \dots - b_{im}\boldsymbol{s}_{nm}$. Using (3.1), the left-hand side of (3.2) coincides with $G(\boldsymbol{s}_n; x_n - 1) \cdot (x_n - 1)^w / \prod_{i\in I}(1 - x_n^{e_{in}})$ and since the left factor of the latter expression converges to $G(\boldsymbol{s}_\infty; 0) \in \mathbf{Q}_\ell^\times$ for $n \to \infty$, the quotient on the right converges to an element of $\mathbf{Q}_\ell$. As $|x_n^e - 1|_\ell = |e|_\ell \cdot |x_n - 1|_\ell$ for $e \in \mathbf{Z}_\ell$ and $\lim_{n\to\infty} e_{in} \neq 0$ for $i \in I$, this easily implies $w \geqslant \# I$ whence $W \in \mathbf{M}[X, Y_1, \dots, Y_m]$. ♦

# 4 Application: functional equations and uniformity

Let $\mathsf{A}$ be a possibly non-associative $\mathbf{Z}$-algebra whose underlying $\mathbf{Z}$-module has finite rank $d$. In an influential paper, du Sautoy and Grunewald [6] established the existence of schemes $V_1, \dots, V_r$ and $W_1, \dots, W_r \in \mathbf{Q}(X, Y)$ such that for almost all primes $p$,

$$\zeta^{\leqslant}_{\mathsf{A}\otimes\mathbf{Z}_p}(s) = \sum_{i=1}^r \# V_i(\mathbf{F}_p) \cdot W_i(p, p^{-s}). \tag{4.1}$$

Voll [20, Thm A] established, for almost all $p$, the functional equation

$$\zeta^{\leqslant}_{\mathsf{A}\otimes\mathbf{Z}_p}(s)\big|_{p\to p^{-1}} = (-1)^d p^{\binom{d}{2} - ds} \cdot \zeta^{\leqslant}_{\mathsf{A}\otimes\mathbf{Z}_p}(s), \tag{4.2}$$

where the operation of "inverting $p$" is defined with respect to a particular formula (4.1). In this section, we will see that this operation is independent of the chosen $V_i$ and

$W_i$ in (4.1) in the following sense: knowing that (4.2) behaves well under local base extensions, any other formula (4.1) (subject to minor technical constraints) behaves in the same way under inversion of $p$ for almost all $p$.

We begin by recalling the formalism for "inverting primes" from [5] which we then combine with our language of local maps from §2. First, let $U$ be a separated scheme of finite type over a finite field $\mathbf{F}_q$. As explained in [5, §2] and [17, §1.5], there are non-zero $m_1, \dots, m_u \in \mathbf{Z}$ and distinct non-zero $\alpha_1, \dots, \alpha_u \in \bar{\mathbf{Q}}$ such that for each $f \in \mathbf{N}$,

$$\#U(\mathbf{F}_{q^f}) = \sum_{i=1}^{u} m_i \alpha_i^f. \tag{4.3}$$

By [5, Lem. 2], up to permutation, the $(m_i, \alpha_i)$ are uniquely determined by (4.3). As in [17, §1.5], one may thus use (4.3) to unambiguously extend the definition of $\#U(\mathbf{F}_{q^f})$ to arbitrary $f \in \mathbf{Z}$.

**Lemma 4.1.** *Let $U_1, \dots, U_r$ be separated $\mathbf{F}_q$-schemes of finite type. Let $W_1, \dots, W_r \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ be regular at $(q^f, Y_1, \dots, Y_m)$ for all $f \in \mathbf{Z} \setminus \{0\}$. Suppose that $\sum\limits_{i=1}^{r} \#U_i(\mathbf{F}_{q^f}) \cdot W_i(q^f, Y_1, \dots, Y_m) = 0$ for all $f \in \mathbf{N}$. Then this identity extends to all $f \in \mathbf{Z} \setminus \{0\}$.*

*Proof.* As in the proof of Theorem 2.3, we may reduce to the case where each $W_i \in \mathbf{Z}[X]$. The result then follows from [5, Lem. 2] and its corollary, cf. the proof of [5, Lem. 3]. ♦

**Corollary 4.2.** *Let $V_1, \dots, V_r$ be separated $\mathfrak{o}$-schemes of finite type and let $W_1, \dots, W_r \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ each be regular at $(q^f, Y_1, \dots, Y_m)$ for all integers $q > 1$ and all $f \in \mathbf{Z} \setminus \{0\}$. Let $P \subset \mathcal{V}_k$ have density 1 and suppose that for all $v \in P$, we have $\sum\limits_{i=1}^{r} \#V_i(\mathfrak{K}_v) \cdot W_i(q_v, Y_1, \dots, Y_m) = 0$. Then there exists a finite set $S \subset \mathcal{V}_k$ such that for all $v \in \mathcal{V}_k \setminus S$ and all $f \in \mathbf{Z} \setminus \{0\}$, we have $\sum\limits_{i=1}^{r} \#V_i(\mathfrak{K}_v^{(f)}) \cdot W_i(q_v^f, Y_1, \dots, Y_m) = 0$.*

*Proof.* Combine Theorem 2.3 and Lemma 4.1. ♦

Let $V_1, \dots, V_r$ be separated $\mathfrak{o}$-schemes of finite type, $W_1, \dots, W_r \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ be regular at $(q^f, Y_1, \dots, Y_m)$ for integers $q > 1$ and $f \in \mathbf{Z} \setminus \{0\}$. Let $\mathsf{Z}$ be a $k$-local map which is equivalent to $[V_1 \cdot W_1] + \dots + [V_r \cdot W_r]$. For a sufficiently large finite set $S \subset \mathcal{V}_k$,

$$\begin{aligned} \mathsf{Z}_* \colon (\mathcal{V}_k \setminus S) \times (\mathbf{Z} \setminus \{0\}) &\to \mathbf{Q}(Y_1, \dots, Y_m), \\ (v, f) &\mapsto \sum_{i=1}^{r} \#V_i(\mathfrak{K}_v^{(f)}) \cdot W_i(q_v^f, Y_1^{\operatorname{sgn}(f)}, \dots, Y_m^{\operatorname{sgn}(f)}) \end{aligned} \tag{4.4}$$

satisfies $\mathsf{Z}_*(v, f) = \mathsf{Z}(v, f)$ for $v \in \mathcal{V}_k \setminus S$ and $f \in \mathbf{N}$. By Lemma 4.1, $\mathsf{Z}_*$ is determined by the equivalence class of $\mathsf{Z}$ (up to enlarging $S$). Let $\hat{\mathsf{Z}}_*(v, f) := \mathsf{Z}_*(v, f)(q_v^{-fs_1}, \dots, q_v^{-fs_m})$.

**Lemma 4.3** (Cf. [5, Cor. to Thm 4])**.** *Suppose that the $k$-local map $\mathsf{Z}$ is uniformly represented by $W \in \mathbf{Q}(X, Y_1, \dots, Y_m)$ which is regular at each point $(q^f, Y_1, \dots, Y_m)$ for integers $q > 1$ and $f \in \mathbf{Z} \setminus \{0\}$. Then for almost all $v \in \mathcal{V}_k$ and all $f \in \mathbf{N}$, we have $\mathsf{Z}_*(v, -f) = W(q_v^{-f}, Y_1^{-1}, \dots, Y_m^{-1})$ and thus $\hat{\mathsf{Z}}_*(v, -f) = W(q_v^{-f}, q_v^{fs_1}, \dots, q_v^{fs_m})$.*

The zeta functions from §2.2 are known to admit extensions of the form (4.4).

**Theorem 4.4.**

*(i)* ([20, Thm A]) *Let $\mathcal{A}$ be a not necessarily associative $k$-algebra of dimension $d$. Then for almost all $v \in \mathcal{V}_k$ and all $f \in \mathbf{N}$,*

$$\mathsf{Z}_*^{\mathcal{A}}(v, -f) = (-1)^d (q_v^f)^{\binom{d}{2}} Y^d \cdot \mathsf{Z}^{\mathcal{A}}(v, f).$$

*(ii)* ([1, Thm A]) *Let $\mathfrak{g}$ be a perfect Lie $k$-algebra of dimension $d$. Then for almost all $v \in \mathcal{V}_k$ and all $f \in \mathbf{N}$,*

$$\mathsf{Z}_*^{\mathfrak{g},\mathrm{irr}}(v, -f) = q_v^{df} \cdot \mathsf{Z}^{\mathfrak{g},\mathrm{irr}}(v, f).$$

*(iii)* ([19, Thm A]) *Let $\mathbf{G}$ be a unipotent algebraic group over $k$. Let $d$ be the dimension of the (algebraic) derived subgroup of $\mathbf{G}$. Then for almost all $v \in \mathcal{V}_k$ and all $f \in \mathbf{N}$,*

$$\mathsf{Z}_*^{\mathbf{G},\widetilde{\mathrm{irr}}}(v, -f) = q_v^{df} \cdot \mathsf{Z}^{\mathbf{G},\widetilde{\mathrm{irr}}}(v, f).$$

Let (4.1) hold for almost all $p$, where the $W_i$ are regular at $(q^f, Y)$ for integers $q > 1$ and $f \in \mathbf{Z} \setminus \{0\}$. Then Corollary 4.2 and Theorem 4.4(i) show that for almost all $p$,

$$\sum_{i=1}^{r} \#V_i(\mathbf{F}_{p^{-1}}) \cdot W_i(p^{-1}, p^s) = (-1)^d p^{\binom{d}{2} - ds} \cdot \zeta^{\leqslant}_{\mathsf{A}\otimes\mathbf{Z}_p}(s),$$

regardless of whether the $V_i$ and $W_i$ were obtained as in the proof of Theorem 4.4(i) or not. Note that in the uniform case ($r = 1$, $V_1 = \mathrm{Spec}(\mathbf{Z})$), we obtain $W_1(X^{-1}, Y^{-1}) = X^{\binom{d}{2}} Y^d \cdot W_1(X, Y)$. This explains why the various uniform examples of local subalgebra zeta functions computed by Woodward [8, 22] satisfy a functional equation even though the methods he used to compute them differ considerably from Voll's approach.

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