

Section 1.4 Techniques of Integration

To calculate

$$\int_a^b f(x) dx$$

- 1 Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f .
- 2 Evaluate F at the limits of integration a and b ; i.e. calculate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
- 3 Calculate the number $F(b) - F(a)$. This is the definite integral $\int_a^b f(x) dx$.

Of the three steps above, the **first one** is the hard one.

The Indefinite Integral

Definition 13

Let f be a function. The **indefinite integral** of f , written

$$\int f(x) dx$$

is the “general antiderivative” of f . If $F(x)$ is a particular antiderivative of f , then we would write

$$\int f(x) dx = F(x) + C,$$

to indicate that the different antiderivatives of f look like $F(x) + C$, where C may be any constant. (In this context C is often referred to as a **constant of integration**).

Notation

Recall the following notation : if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)|_a^b \text{ or } F(x)|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

Definition 12

Let f be a function. Another function F is called an **antiderivative** of f if the derivative of F is f , i.e. if $F'(x) = f(x)$, for all (relevant) values of the variable x .

So for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about **an** antiderivative of a function or expression rather than **the** antiderivative.

Examples

Example 14

Determine $\int \cos 2x dx$.

Solution: The question is: what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving \cos ? The derivative of $\sin x$ is $\cos x$. A reasonable guess would say that the derivative of $\sin 2x$ might be “something like” $\cos 2x$. By the chain rule, the derivative of $\sin 2x$ is in fact $2 \cos 2x$. So $\sin 2x$ is pretty close but it gives us twice what we want - we should compensate for this by taking $\frac{1}{2} \sin 2x$; its derivative is

$$\frac{1}{2}(2 \cos 2x) = \cos 2x.$$

$$\text{Conclusion: } \int \cos 2x dx = \frac{1}{2} \sin 2x + C$$

Powers of x

Example 15

Determine $\int x^n dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n , we reduce the index by 1 to $n - 1$, and we multiply by the constant n . So

$$\frac{d}{dx} x^n = nx^{n-1}$$

in general. To find an **antiderivative** of x^n we have to reverse this process. This means that the index **increases** by 1 to $n + 1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x} dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x} dx = \ln |x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 16

Determine $\int_0^\pi \sin x + \cos x dx$.

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned} \int_0^\pi \sin x + \cos x dx &= -\cos x + \sin x \Big|_0^\pi \\ &= (-\cos \pi + \sin \pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2. \end{aligned}$$

Note: To determine $\cos \pi$, start at the point $(1, 0)$ and travel counter-clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x -coordinate of the point you are at now is $\cos \pi$, and the y -coordinate is $\sin \pi$.