Powers of x

Example 15

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x} \, dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x} \, dx = \ln |x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 16

Determine $\int_0^{\pi} \sin x + \cos x \, dx.$

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned} \int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x|_0^{\pi} \\ &= (-\cos\pi + \sin\pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2. \end{aligned}$$

Note: To determine $\cos\pi$, start at the point $(1, 0)$ and travel counter–clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos\pi$, and the y-coordinate is $\sin\pi$.

3.4.1 Substitution - Reversing the Chain Rule

The Chain Rule of Differentation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something") is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x\cos x^2.$$

Equivalently

$$\int 2x\cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x\cos x^2$ back to $\sin x^2$.

How Substitution Works

Example 17

Determine $\int 2x\sqrt{x^2+1}\,dx$.

Solution Notice that the integrand involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.

1. Introduce the notation u and set $u = x^2+1$.
2. Note $\frac{du}{dx} = 2x$; rewrite this as $du = 2x\,dx$.
3. Then
$$\int 2x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}(2x\,dx) = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C.$$
4. So
$$\int 2x\sqrt{x^2+1}\,dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

Substitution and definite integrals

Example 18

Determine $\int_0^{\pi} \cos^3 x \sin x\,dx$ *(from 2015 Summer paper)*

Solution: Write $u = \cos x$. Then

$$\frac{du}{dx} = -\sin x,\ du = -\sin x\,dx,\ \ \sin x\,dx = -du.$$

Change variables: $\int_0^{\pi}\cos^3 x \sin x\,dx = -\int_{x=0}^{x=\pi} u^3\,du$. Limits of integration: When $x=0$, $u = \cos x = \cos 0 = 1$. When $x = \pi$, $u = \cos x = \cos\pi = -1$. Our integral becomes:

$$\int_{u=1}^{u=-1} u^3\,du = \left.\frac{u^4}{4}\right|_{u=-1}^{u=1} = \frac{1}{4} - \frac{(-1)^4}{4} = 0.$$

Substitution and Definite Integrals - more examples

Example 19

Evaluate $\int_0^1 \frac{5r}{(4+r^2)^2}\,dr$.

Solution To find an antiderivative, let $u = 4+r^2$.
Then $\frac{du}{dr} = 2r,\ du = 2r\,dr;\ 5r\,dr = \frac{5}{2}du$.
So
$$\int \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\int u^{-2}\,du.$$
Thus
$$\int \frac{5r}{(4+r^2)^2}\,dr = -\frac{5}{2}\times\frac{1}{u} + C,$$
and we need to evaluate $-\frac{5}{2}\times\frac{1}{u}$ at $r=0$ and at $r=1$. We have two choices.

Two Choices

1. Write $u = 4+r^2$ to obtain

$$\begin{aligned}\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2}\times\left.\frac{1}{4+r^2}\right|_{r=0}^{r=1}\\ &= -\frac{5}{2}\times\frac{1}{4+1^2} - \left(-\frac{5}{2}\times\frac{1}{4+0^2}\right)\\ &= -\frac{5}{2}\times\frac{1}{5} + \frac{5}{2}\times\frac{1}{4}\\ &= \frac{1}{8}.\end{aligned}$$

. . . Alternatively

2. Alternatively, write the antiderivative as $-\frac{5}{2} \times \frac{1}{u}$ and replace the limits of integration with the corresponding values of u.
   When $r = 0$ we have $u = 4 + 0^2 = 4$.
   When $r = 1$ we have $u = 4 + 1^2 = 5$.
   Thus

$$\begin{aligned}\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2} \times \frac{1}{u}\Big|_{u=4}^{u=5} \\ &= -\frac{5}{2} \times \frac{1}{5} - \left(-\frac{5}{2} \times \frac{1}{4}\right) \\ &= \frac{1}{8}.\end{aligned}$$

From Summer Exam 2013

Example 20

Determine

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx.$$

Now write $u = \sqrt{x} + 1$. Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} \Longrightarrow \frac{1}{\sqrt{x}}\,dx = 2du.$

Then

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int_{x=1}^{x=4} \frac{2}{u}\,du = \int_{u=2}^{u=3} \frac{2}{u}\,du = 2\ln u|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2\ln\frac{3}{2}.\end{aligned}$$

From Summer Exam 2013

Example

Determine

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Note on the exam: This question was not answered well. It was not intended to be particularly difficult or tricky. Only about five people in the whole class answered it correctly. Many candidates made very fundamental and serious errors in algebra before attempting the integration, for example rewriting $\frac{1}{x+\sqrt{x}}$ as $\frac{1}{x} + \frac{1}{\sqrt{x}}$. No credit could be awarded in such a case since the error rendered the rest of the question meaningless. So BE CAREFUL.

More Examples

Example 21

Determine $\int (1-\cos t)^2 \sin t\,dt$

Question: *How do we know what expression to extract and refer to as u?*
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.
There is no easy answer but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1 - \cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1 - \cos t$ effective for this problem.

NOTE: There are more examples of the substitution technique in the lecture notes.