

Section 1.4.3 : Partial Fraction Expansions

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4 \, dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

and that

$$\int \frac{1}{x^n} \, dx = -\frac{1}{n-1} \frac{1}{x^{n-1}} + C,$$

for $n > 1$.

This section is about integrating **rational functions**; i.e. quotients in which the numerator and denominator are both polynomials.

Adding Symbolic Fractions

Remark: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a **common denominator** and write

$$\begin{aligned} \frac{2}{x+3} + \frac{1}{x+4} &= \frac{2(x+4)}{(x+3)(x+4)} + \frac{1(x+3)}{(x+3)(x+4)} \\ &= \frac{2(x+4) + 1(x+3)}{(x+3)(x+4)} = \frac{3x+11}{(x+3)(x+4)}. \end{aligned}$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$

and asked to rewrite it in the form $\frac{A}{x+3} + \frac{B}{x+4}$, for *numbers* A and B . How would we do it?

Another Question Why would we want to do such a thing?

The Partial Fraction Expansion

Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x + 4A + 3B}{(x+3)(x+4)}.$$

This means $3x+11 = (A+B)x + 4A + 3B$ for all x , which means

$$A+B=3, \text{ and } 4A+3B=11.$$

Thus $B=1$ and $A=2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

An Alternative Method

We want

$$3x+11 = A(x+4) + B(x+3),$$

for **all** real numbers x . If this statement is true for all x , then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12+11 = A(0) + B(-1) \implies B=1.$$

Setting $x = -3$ gives

$$-9+11 = A(1) + B(0) \implies A=2.$$

Thus

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Integration using partial fractions

Example 28

Determine $\int \frac{3x+11}{(x+3)(x+4)} dx$.

Solution : Write

$$\int \frac{3x+11}{(x+3)(x+4)} dx = \int \frac{2}{x+3} dx + \int \frac{1}{x+4} dx$$

Then

$$\int \frac{3x+11}{(x+3)(x+4)} dx = 2 \ln|x+3| + \ln|x+4| + C = \ln(x+3)^2 + \ln|x+4| + C.$$

A Harder Example

Example 30

Determine $\int \frac{x+1}{(2x+1)^2(x-2)} dx$.

Solution: In this case the denominator has a **repeated** linear factor $2x+1$. It is necessary to include both $\frac{A}{2x+1}$ and $\frac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2) + B(x-2) + C(2x+1)^2}{(2x+1)^2(x-2)}.$$

and so

$$x+1 = A(2x+1)(x-2) + B(x-2) + C(2x+1)^2.$$

Partial fractions with long division

Example 29

Determine $\int \frac{x^3+3x+2}{x+1} dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.

We find that the quotient is $x^2 - x + 4$ and the remainder is -2 . Hence

$$\frac{x^3+3x+2}{x+1} = x^2 - x + 4 + \frac{-2}{x+1}.$$

Thus

$$\begin{aligned} \int \frac{x^3+3x+2}{x+1} dx &= \int x^2 - x + 4 dx - 2 \int \frac{1}{x+1} dx \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + 4x - 2 \ln|x+1| + C. \end{aligned}$$

A Harder Example

$$x=2: 3 = C(5)^2 \quad C = \frac{3}{25}$$

$$x=-\frac{1}{2}: \frac{1}{2} = B\left(-\frac{5}{2}\right) \quad B = -\frac{1}{5}$$

$$x=0: 1 = A(1)(-2) + B(-2) + C(1)^2 \quad A = -\frac{6}{25}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1} + \frac{-1/5}{(2x+1)^2} + \frac{3/25}{x-2}$$

and

$$\begin{aligned} \int \frac{x+1}{(2x+1)^2(x-2)} dx &= -\frac{6}{25} \int \frac{1}{2x+1} dx - \frac{1}{5} \int \frac{1}{(2x+1)^2} dx \\ &\quad + \frac{3}{25} \int \frac{1}{x-2} dx. \end{aligned}$$

A Harder Example

Call the three integrals on the right above I_1 , I_2 , I_3 respectively.

$$\blacksquare I_1 : \int \frac{1}{2x+1} dx = \frac{1}{2} \ln |2x+1| (+C_1).$$

$$\blacksquare I_2 : \int \frac{1}{(2x+1)^2} dx = -\frac{1}{2(2x+1)} (+C_2).$$

$$\blacksquare I_3 : \int \frac{1}{x-2} dx = \ln |x-2| (+C_3).$$

Thus

$$\int \frac{x+1}{(2x+1)^2(x-2)} dx = -\frac{3}{25} \ln |2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25} \ln |x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
 - direct methods;
 - suitably chosen substitutions;
 - integration by parts;
 - partial fraction expansions.