Recall from Section 2.2 that

- The cardinality of a finite set is defined as the number of elements in it.
- Two sets A and B have the same cardinality if (and only if) it is possible to match each element of A to an element of B in such a way that every element of each set has exactly one "partner" in the other set.

In the case of finite sets, the second point above might seem to be overcomplicating the issue, since we can tell if two finite sets have the same cardinality by just counting their elements and noting that they have the same number.

The case of infinite sets

> Two sets have the same cardinality if (and only if) it is possible to match each element of A to an element of B in such a way that every element of each set has exactly one "partner" in the other set.

The notion of bijective correspondence is emphasized for two reasons.

- It is occasionally possible to establish that two finite sets are in bijective correspondence without knowing the cardinality of either of them.
- We can't count the number of elements in an infinite set. However, for a given pair of infinite sets, we could possibly show that it is or isn't possible to construct a bijective correspondence between them.

Infinite sets having the same cardinality

Definition

Suppose that A and B are sets (finite or infinite). We say that A and B have the same cardinality (written $|A| = |B|$) if a bijective correspondence exists between A and B.

In other words, A and B have the same cardinality if it's possible to match each element of A to a different element of B in such a way that every element of both sets is matched exactly once. In order to say that A and B have different cardinalities we need to establish that it's impossible to match up their elements with a bijective correspondence. If A and B are infinite sets, showing that such a thing is *impossible* can be a formidable challenge.

$\mathbb{N}$ and $\mathbb{Z}$

Definition

The set $\mathbb{N}$ of natural numbers ("counting numbers") consists of all the positive integers. $\mathbb{N} = \{1, 2, 3, \dots\}$.

Example

Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

We need to fill the right-hand column of the table below with the integers *in some order*, in such a way that each integer appears there exactly once.

$\mathbb{N}$		$\mathbb{Z}$
1	$\longleftrightarrow$	?
2	$\longleftrightarrow$	?
3	$\longleftrightarrow$	?
4	$\longleftrightarrow$	?
$\vdots$	$\longleftrightarrow$	$\vdots$

Bijective correspondence between $\mathbb{N}$ and $\mathbb{Z}$

So we need to list all the integers on the right hand side, in such a way that every integer appears once. Just following the natural order on the integers won't work, because then there is no first entry for our list.

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ \hline 1 & \longleftrightarrow & ? \\ 2 & \longleftrightarrow & ? \\ 3 & \longleftrightarrow & ? \\ 4 & \longleftrightarrow & ? \\ \vdots & \longleftrightarrow & \vdots \end{array}$$

Starting at a particular integer like 0 and then following the natural order won't work, because then we will never get (for example) any negative integers in our list.

$$\begin{array}{ccc} \mathbb{N} & & \mathbb{Z} \\ \hline 1 & \longleftrightarrow & ? \\ 2 & \longleftrightarrow & ? \\ 3 & \longleftrightarrow & ? \\ \vdots & \longleftrightarrow & \vdots \end{array}$$

Bijective correspondence $\mathbb{N} \longleftrightarrow \mathbb{Z}$

We can start with 0, then list 1 and then -1, then 2 and then -2, then 3 and then -3 and so on. This is a systematic way of writing out all the integers, in which each appears exactly once. Our table becomes

$\mathbb{N}$		$\mathbb{Z}$
1	$\longleftrightarrow$	0
2	$\longleftrightarrow$	1
3	$\longleftrightarrow$	-1
4	$\longleftrightarrow$	2
5	$\longleftrightarrow$	-2
6	$\longleftrightarrow$	3
$\vdots$	$\longleftrightarrow$	$\vdots$

Exercise 39

What integer corresponds to the natural number 22 in the list?
In what position does the integer -63 appear?

A more explicit version

If we want to be fully explicit about how this bijective correspondence works, we can even give a formula for the integer that is matched to each natural number. The correspondence above describes a bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ given for $n \in \mathbb{N}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if} \quad n \text{ is even} \\ \\ -\left(\frac{n-1}{2}\right) & \text{if} \quad n \text{ is odd} \end{cases}$$

As well as understanding this example at the informal/intuitive level suggested by the picture above, think about the formula above, and satisfy yourself that it does indeed descibe a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Curiosities of Infinite Sets

The example above demonstrates a curious thing that can happen when considering cardinalities of infinite sets. The set $\mathbb{N}$ of natural numbers is a proper subset of the the set $\mathbb{Z}$ of integers (this means that every natural number is an integer, but the natural numbers do not account for all the integers).

Yet we have just shown that $\mathbb{N}$ and $\mathbb{Z}$ can be put in bijective correspondence. So it is possible for an infinite set to be in bijective correspondence with a proper subset of itself, and hence to have the same cardinality as a proper subset of itself.

This can't happen for finite sets (why?).

Countably Infinite Sets

Putting an infinite set in bijective correspondence with $\mathbb{N}$ amounts to providing a robust and unambiguous scheme or instruction for listing all its elements starting with a first, then a second, third, etc., in such a way that it can be seen that every element of the set will appear exactly once in the list.

Definition

A set is called countably infinite (or denumerable) if it can be put in bijective correspondence with the set of natural numbers. A set is called countable if it is either finite or countably infinite.

Basically, an infinite set is countable if its elements can be listed in an inclusive and organised way. “Listable” might be a better word, but it is not really used.

Countability of $\mathbb{Q}$

Thus the sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality. Maybe this is not so surprising, because these sets have a strong geometric resemblance as sets of points on the number line.

What is more surprising is that $\mathbb{N}$ (and hence $\mathbb{Z}$) has the same cardinality as the set $\mathbb{Q}$ of all rational numbers. These sets do not resemble each other much geometrically. The natural numbers are sparse and evenly spaced, whereas the rational numbers are densely packed into the number line.

Nevertheless, as the following construction shows, $\mathbb{Q}$ is a countable set.

$\mathbb{Q}$ is countable

We need to show that the rational numbers can be organized into a numbered list in a systematic way that includes all of them. Such a list is a one-to-correspondence with the set $\mathbb{N}$ of natural numbers.

Start with the following array of fractions.

$$
\begin{array}{ccccccccc}
\ldots & -3/1 & -2/1 & -1/1 & 0/1 & 1/1 & 2/1 & 3/1 & \ldots \\
\ldots & -3/2 & -2/2 & -1/2 & 0/2 & 1/2 & 2/2 & 3/2 & \ldots \\
\ldots & -3/3 & -2/3 & -1/3 & 0/3 & 1/3 & 2/3 & 3/3 & \ldots \\
\ldots & -3/4 & -2/4 & -1/4 & 0/4 & 1/4 & 2/4 & 3/4 & \ldots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

$\mathbb{Q}$ is countable

We need to show that the rational numbers can be organized into a numbered list in a systematic way that includes all of them. Such a list is a one-to-correspondence with the set $\mathbb{N}$ of natural numbers.
Construct a path through the whole array :

$$
\begin{array}{ccccccccccccccc}
\cdots & -3/1 & & -2/1 & \leftarrow & -1/1 & & 0/1 & \rightarrow & 1/1 & & 2/1 & \rightarrow & 3/1 & \cdots \\
 & \uparrow & & \downarrow & & \uparrow & & & & \downarrow & & \uparrow & & \downarrow & \\
\cdots & -3/2 & & -2/2 & & -1/2 & \leftarrow & 0/2 & \leftarrow & 1/2 & & 2/2 & & 3/2 & \cdots \\
 & \uparrow & & \downarrow & & & & & & & & \uparrow & & \downarrow & \\
\cdots & -3/3 & & -2/3 & \rightarrow & -1/3 & \rightarrow & 0/3 & \rightarrow & 1/3 & \rightarrow & 2/3 & & 3/3 & \cdots \\
 & \uparrow & & & & & & & & & & & & \downarrow & \\
\cdots & -3/4 & \leftarrow & -2/4 & \leftarrow & -1/4 & \leftarrow & 0/4 & \leftarrow & 1/4 & \leftarrow & 2/4 & \leftarrow & 3/4 & \cdots \\
 & & & & & & & & & & & & & & \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

$\mathbb{Q}$ is countable

In these fractions, the numerators increase through all the integers as we travel along the rows, and the denominators increase through all the natural numbers as we travel downwards through the columns.
Every rational number occurs somewhere in the array.

This path determines a listing of all the fractions in the array, that starts as follows

$0/1,\ 1/1,\ 1/2,\ 0/2,\ -1/2,\ -1/1,\ -2/1,\ -2/2,\ -2/3,\ -1/3,\ 0/3,\ 1/3,\ 2/3,\ 2/2,\ 2/1,$

$\mathbb{Q}$ is countable

$$0/1,\ 1/1,\ 1/2,\ 0/2,\ -1/2,\ -1/1,\ -2/1,\ -2/2,\ -2/3,\ -1/3,\ 0/3,\ 1/3,$$

$$2/3,\ 2/2,\ 2/1,\ 3/1,\ 3/2,\ 3/3,\ 3/4,\ \dots$$

What this construction demonstrates is a bijective correspondence between the set $\mathbb{N}$ of natural numbers and the set of all fractions in our array.
This is not (exactly) a bijective correspondence between $\mathbb{N}$ and $\mathbb{Q}$.

Exercise 40

Why not? (Think about this before reading on.)

The reason why not is that every rational number appears many times in our array.

$\mathbb{Q}$ is countable

In order to get a bijective correspondence between $\mathbb{N}$ and $\mathbb{Q}$, construct a list of all the rational numbers from the array as above, but whenever a rational number is encountered that has already appeared, leave it out. Our list will begin

$$0/1,\ 1/1,\ 1/2,\ -1/2,\ -1/1,\ -2/1,\ -2/3,\ -1/3,\ 1/3,\ 2/3,\ 2/1,$$

$$3/1,\ 3/2,\ 3/4,\ \ldots$$

We conclude that the rational numbers are countable.

Note : Unlike our one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$, in this case we cannot write down a simple formula to tell us what rational number will be Item 34 on our list (i.e. corresponds to the natural number 34) or where in our list the rational number $292/53$ will appear.

An open interval

In our next example we show that the set of all the real numbers has the same cardinality as an open interval on the real line.

Example

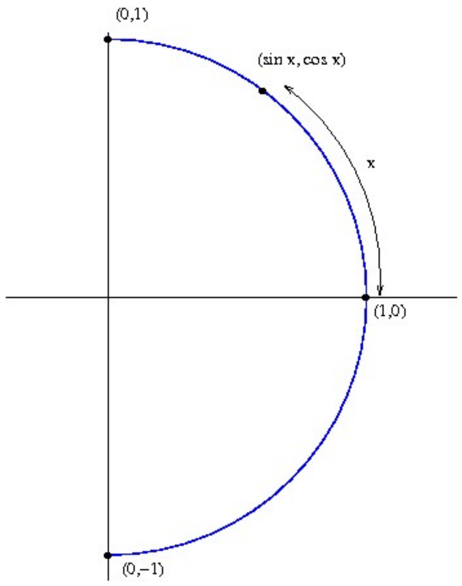
Show that $\mathbb{R}$ *has the same cardinality as the open interval* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

In order to do this we have to establish a bijective correspondence between the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.
An example of a function that provides us with such a bijective correspondence is familiar from calculus/trigonometry.

$(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$

Recall that for a number x in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\tan x$ is defined as follows: travel from $(1, 0)$ a distance $|x|$ along the circumference of the unit circle, anti-clockwise if x is positive and clockwise if x is negative. We arrive at a point which is in the right-hand side of the unit circle.

Now $\tan x$ is the slope of the line that connects the origin to this point (whose y and x-coordinates are $\sin x$ and $\cos x$ respectively).



$\tan x$ gives a bijection

Now $\tan 0 = 0$, and as x increases from 0 towards $\frac{\pi}{2}$, the line segment in question rotates about the origin into the first quadrant, its slope increases continuously from zero, without limit as x approaches $\frac{\pi}{2}$. So every positive real number is the tan of exactly one x in the range $(0, \frac{\pi}{2})$.

For the same reason, the values of $\tan x$ include every negative real number exactly once as x runs between 0 and $-\frac{\pi}{2}$.

$\tan x$ gives a bijection

Thus for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the correspondence

$$x \longleftrightarrow \tan x$$

establishes a bijection between the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.

We conclude that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same cardinality as $\mathbb{R}$.

Note: This assertion is unrelated to the concept of countability discussed earlier.

Some Remarks

1. We don't know yet if $\mathbb{R}$ (or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) has the same cardinality as $\mathbb{N}$ - we don't know if $\mathbb{R}$ is countable.
2. The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ may seem like an odd choice for an example like this. However, note that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is in bijective correspondence with the interval $(-1, 1)$, via the function that just multiplies everything by $\frac{2}{\pi}$.

Exercise 41

Show that the the open interval $(0, 1)$ *has the same cardinality as*

1. *The open interval* $(-1, 1)$
2. *The open interval* $(1, 2)$
3. *The open interval* $(2, 6)$.

Bounded and unbounded subsets of $\mathbb{R}$

Basically, a subset X of $\mathbb{R}$ is bounded if, on the number line, its elements do not extend indefinitely to the left or right. In other words there exist real numbers a and b with $a < b$, for which all the points of X are in the interval (a, b).

Definition

Let X be a subset of $\mathbb{R}$. Then X is bounded below if there exists a real number a with $a < x$ for all elements x of X. (Note that a need not belong to X here).
The set X is bounded above if there exists a real number b with $x < b$ for elements x of X. (Note that b need not belong to X here).
The set X is bounded if it is bounded above and bounded below (otherwise it's unbounded).

Bounded and unbounded sets

Example

1. $\mathbb{Q}$ *is unbounded.*
2. $\mathbb{N}$ *is bounded below but not above.*
3. $(0,1)$, $[0,1]$, $[2,100]$ *are bounded.*
4. $\{\cos x : x \in \mathbb{R}\}$ *is bounded, since* $\cos x$ *can only have values between* -1 *and* 1.
5. *All finite subsets of* $\mathbb{R}$ *are bounded, and some infinite subsets are.*

Remark: The last example (involving $\tan x$) shows that it is possible for a bounded subset of $\mathbb{R}$ to have the same cardinality as the full set $\mathbb{R}$ of real numbers.

Learning outcomes for Section 2.3

This section contains some very challenging concepts. You will probably need to invest some serious intellectual effort in order to arrive at a good understanding of them. This is an effort worth making as it has the potential to really expand your view of what mathematics is about.

After studying this section you should be able to

- Discuss the concept of bijective correspondence for infinite sets;
- Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality by exhibiting a bijective correspondence between them;
- Explain what is meant by a *countable* set and show that $\mathbb{Q}$ is countable;
- Exhibit a bijective correspondence between $\mathbb{R}$ and the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence show that $\mathbb{R}$ has the same cardinality as the interval (a, b) for any real numbers a and b with $a < b$.