

The rational numbers and real numbers are closely related.

- The set  $\mathbb{Q}$  of rational numbers is countable and the set  $\mathbb{R}$  of real numbers is not, and in this sense there are many more real numbers than rational numbers.
- However,  $\mathbb{Q}$  is “dense” in  $\mathbb{R}$ . This means that every interval of the real number line, no matter how short, contains infinitely many rational numbers. This statement has a practical impact as well, which we use all the time.

#### Lemma 43

*Every real number (whether rational or not) can be approximated by a rational number with a level of accuracy as high as we like.*

- 3 is a rational approximation for  $\pi$ .
- 3.1 is a closer one.
- 3.14 is closer again.
- 3.14159 is closer still.
- 3.1415926535 is even closer than that,

and we can keep improving on this by truncating the decimal expansion of  $\pi$  at later and later stages.

If we want a rational approximation that differs from the true value of  $\pi$  by less than  $10^{-20}$  we can truncate the decimal approximation of  $\pi$  at the 21st digit after the decimal point. This is what is meant by “a level of accuracy as high as we like” in the statement of the lemma.

## Notes

- 1 The fact that all real numbers can be approximated with arbitrary closeness by rational numbers is used all the time in everyday life. Computers basically don't deal with all the real numbers or even with all the rational numbers, but with some specified level of precision. They really work with a subset of the rational numbers.

- 2 The **sequence**

3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, ...

is a list of numbers that are steadily approaching  $\pi$ . The terms in this sequence are increasing and they are approaching  $\pi$ . We say that this sequence **converges** to  $\pi$  and we will investigate the concept of convergent sequences in Chapter 3.

- 3 We haven't looked yet at the question of how the numbers in the above sequence can be calculated, i.e. how we can get our hands on better and better approximations to the value of the irrational number  $\pi$ . That's another thing that we will look at in Chapter 3.

## Upper and Lower Bounds

The goal of this last section of Chapter 2 is to pinpoint one essential property of subsets of  $\mathbb{R}$  that is not shared by subsets of  $\mathbb{Z}$  or of  $\mathbb{Q}$ . We need a few definitions and some terminology in order to describe this.

#### Definition 44

Let  $S$  be a subset of  $\mathbb{R}$ . An element  $b$  of  $\mathbb{R}$  is an **upper bound** for  $S$  if  $x \leq b$  for all  $x \in S$ . An element  $a$  of  $\mathbb{R}$  is a **lower bound** for  $S$  if  $a \leq x$  for all  $x \in S$ .

So an upper bound for  $S$  is a number that is to the right of all elements of  $S$  on the real line, and a lower bound for  $S$  is a number that is to the left of all points of  $S$  on the real line. Note that if  $b$  is an upper bound for  $S$ , then so is every number  $b'$  with  $b < b'$ . If  $a$  is a lower bound for  $S$  then so is every number  $a'$  with  $a' < a$ . So if  $S$  has an upper bound at all it has infinitely many upper bounds, and if  $S$  has a lower bound at all it has infinitely many lower bounds.

## Upper and Lower Bounds

### Definition 45

Let  $S$  be a subset of  $\mathbb{R}$ . An element  $b$  of  $\mathbb{R}$  is an **upper bound** for  $S$  if  $x \leq b$  for all  $x \in S$ . An element  $a$  of  $\mathbb{R}$  is a **lower bound** for  $S$  if  $a \leq x$  for all  $x \in S$ .

Recall that

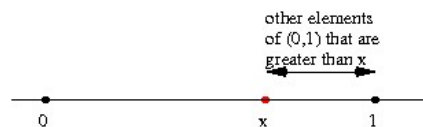
- $S$  is **bounded above** if it has an upper bound,
- $S$  is **bounded below** if it has a lower bound,
- $S$  is **bounded** if it is bounded both above and below.

In this section we are mostly interested in sets that are bounded on at least one side.

## Not every set has a maximum element

There are basically two reasons why a subset  $S$  of  $\mathbb{R}$  might **fail** to have a maximum element. First,  $S$  **might not be bounded above** - then it certainly won't have a maximum element.

Secondly,  $S$  might be bounded above, but **might not contain an element that is an upper bound for itself**. Take for example an **open interval** like  $(0, 1)$ . This set is certainly bounded above. However, take any element  $x$  of  $(0, 1)$ . Then  $x$  is a real number that is strictly greater than 0 and strictly less than 1. Between  $x$  and 1 there are more real numbers all of which belong to  $(0, 1)$  and are greater than  $x$ . So  $x$  cannot be an upper bound for the interval  $(0, 1)$ .



## Maximum and minimum elements

### Definition 46

Let  $S$  be a subset of  $\mathbb{R}$ . If there is a number  $m$  that is both an **element** of  $S$  and an **upper bound** for  $S$ , then  $m$  is called the **maximum element** of  $S$  and denoted  $\max(S)$ .

If there is a number  $l$  that is both an **element** of  $S$  and a **lower bound** for  $S$ , then  $l$  is called the **minimum element** of  $S$  and denoted  $\min(S)$ .

### Notes

A set can have at most one maximum (or minimum) element.

Pictorially, on the number line, the maximum element of  $S$  is the rightmost point that belongs to  $S$ , if such a point exists. The minimum element of  $S$  is the leftmost point on the number line that belongs to  $S$ , if such a point exists.

## Maximum and Minimum Elements

An open interval like  $(0, 1)$ , although it is bounded, has no maximum element and no minimum element.

An example of a subset of  $\mathbb{R}$  that *does* have a maximum and a minimum element is a **closed** interval like  $[2, 3]$ . The minimum element of  $[2, 3]$  is 2 and the maximum element is 3.

**Remark :** Every **finite** subset of  $\mathbb{R}$  has a maximum element and a minimum element.

## Supremum and Infimum

For bounded subsets of  $\mathbb{R}$ , there are notions called the **supremum** and **infimum** that are closely related to maximum and minimum. Every subset of  $\mathbb{R}$  that is bounded above has a supremum and every subset of  $\mathbb{R}$  that is bounded below has an infimum.

### Definition 47 (The Axiom of Completeness for $\mathbb{R}$ )

Let  $S$  be a subset of  $\mathbb{R}$  that is bounded above. Then the set of **all upper bounds** for  $S$  has a minimum element. This number is called the **supremum** of  $S$  and denoted  $\sup(S)$ .

Let  $S$  be a subset of  $\mathbb{R}$  that is bounded below. Then the set of **all lower bounds** for  $S$  has a maximum element. This number is called the **infimum** of  $S$  and denoted  $\inf(S)$ .

### Notes

- 1 The **supremum** of  $S$  is also called the **least upper bound (lub)** of  $S$ .
- 2 The **infimum** of  $S$  is also called the **greatest lower bound (glb)** of  $S$ .

## The Axiom of Completeness

The definition above is simultaneously a definition of the terms **supremum** and **infimum** and a statement of the **Axiom of Completeness** for the real numbers.

To see why this statement says something special about the real numbers, temporarily imagine that the only number system available to us is  $\mathbb{Q}$ , the set of rational numbers. Look at the set

$$S := \{x \in \mathbb{Q} : x^2 < 2\}.$$

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So  $S$  consists of all those rational numbers whose square is less than 2. It is bounded above, for example by 2.

The positive elements of  $S$  are all those positive rational numbers that are less than the real number  $\sqrt{2}$ .

**Claim:**  $S$  does not have a least upper bound in  $\mathbb{Q}$ .

To see this, suppose that  $x$  is a rational number that is a candidate for being the least upper bound of  $S$  in  $\mathbb{R}$ .

- If  $x^2 < 2$ , then there is a gap in the number line between  $x$  and  $\sqrt{2}$ , and in this gap are rational numbers that are greater than  $x$  but still less than  $\sqrt{2}$ . So  $x$  is not an upper bound of  $S$ .
- If  $x^2 > 2$ , then there is a gap in the number line between  $\sqrt{2}$  and  $x$ , and in this gap are rational numbers that are still upper bounds of  $S$  but are less than  $x$ .

$$S := \{x \in \mathbb{Q} : x^2 < 2\}$$

If we consider the same set  $S$  as a subset of  $\mathbb{R}$ , we can see that  $\sqrt{2}$  is the supremum of  $S$  in  $\mathbb{R}$  (and  $-\sqrt{2}$  is the infimum of  $S$  in  $\mathbb{R}$ ).

This example demonstrates that the **Axiom of Completeness** does not hold for  $\mathbb{Q}$ , i.e. a bounded subset of  $\mathbb{Q}$  need not have a **supremum** in  $\mathbb{Q}$  or an **infimum** in  $\mathbb{Q}$ .

Question 48

Let  $S = \left\{ \frac{2n+4}{3n} : n \in \mathbb{Z}, n \geq 1 \right\}$ .

- 1 List four elements of  $S$ .
- 2 Identify, with explanation, the maximum element of  $S$ .
- 3 Show that  $S$  has no minimum element, and determine the infimum of  $S$ .

After studying this section you should be able to

- State what it means for a subset of  $\mathbb{R}$  to be *bounded* (or *bounded above* or *bounded below*).
- Define the terms maximum, minimum, supremum and infimum and explain the connections and differences between them.
- State the Axiom of Completeness.
- Determine whether a set presented like the one in the problem above is bounded (above and/or below) or not and identify its maximum/minimum/infimum/supremum as appropriate, with explanation.