

## Section 3.2 : Sequences

**Note:** Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

### Definition 53

A **sequence** is an infinite ordered list

$$a_1, a_2, a_3, \dots$$

- The items in list  $a_1, a_2$  etc. are called **terms** (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence  $a_1, a_2, \dots$  can be denoted by  $(a_n)$  or by  $(a_n)_{n=1}^{\infty}$ .
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

## Visualising a sequence

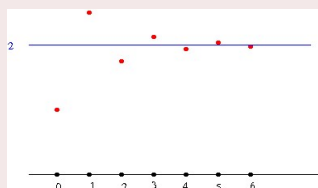
One way of visualizing a sequence is to consider it as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

### Example 54

$(2 + (-1)^n 2^{1-n})_{n=1}^{\infty}$ . Write  $a_n = 2 + (-1)^n 2^{1-n}$ . Then

$$a_1 = 2 - 2^0 = 1, \quad a_2 = 2 + 2^{-1} = \frac{5}{2}, \quad a_3 = 2 - 2^{-2} = \frac{7}{4}, \quad a_4 = 2 + 2^{-3} = \frac{17}{8}.$$

Graphical representation of  $(a_n)$ :



## A Few Examples

1  $((-1)^n + 1)_{n=1}^{\infty}$  :  $a_n = (-1)^n + 1$   
 $a_1 = -1 + 1 = 0, \quad a_2 = (-1)^2 + 1 = 2, \quad a_3 = (-1)^3 + 1 = 0, \dots$

$$0, 2, 0, 2, 0, 2, \dots$$

2  $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$  :  $a_n = \sin(\frac{n\pi}{2})$   
 $a_1 = \sin(\frac{\pi}{2}) = 1, \quad a_2 = \sin(\pi) = 0, \quad a_3 = \sin(\frac{3\pi}{2}) = -1, \quad a_4 = \sin(2\pi) = 0, \dots$

$$1, 0, -1, 0, 1, 0, -1, 0, \dots$$

3  $(\frac{1}{n} \sin(\frac{n\pi}{2}))_{n=1}^{\infty}$  :  $a_n = \frac{1}{n} \sin(\frac{n\pi}{2})$   
 $a_1 = \sin(\frac{\pi}{2}) = 1, \quad a_2 = \frac{1}{2} \sin(\pi) = 0, \quad a_3 = \frac{1}{3} \sin(\frac{3\pi}{2}) = -\frac{1}{3}, \quad a_4 = \frac{1}{4} \sin(2\pi) = 0, \dots$

$$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \dots$$

## The sequence $(2 + (-1)^n \frac{1}{2^{n-1}})_{n=1}^{\infty}$

As  $n$  gets very large the positive number  $\frac{1}{2^{n-1}}$  gets very small. By taking  $n$  as large as we like, we can make  $\frac{1}{2^{n-1}}$  as small as we like.

Hence for very large values of  $n$ , the number  $2 + (-1)^n \frac{1}{2^{n-1}}$  is very close to 2. By taking  $n$  as large as we like, we can make this number as close to 2 as we like.

We say that the sequence **converges** to 2, or that 2 is the **limit** of the sequence, and write

$$\lim_{n \rightarrow \infty} \left( 2 + (-1)^n \frac{1}{2^{n-1}} \right) = 2.$$

**Note:** Because  $(-1)^n$  is alternately positive and negative as  $n$  runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

## Convergence of a sequence : “official” definitions

### Definition 55

The sequence  $(a_n)$  **converges** to the number  $L$  (or has **limit**  $L$ ) if for every positive real number  $\varepsilon$  (no matter how small) there exists a natural number  $N$  with the property that the term  $a_n$  of the sequence is within  $\varepsilon$  of  $L$  for all terms  $a_n$  beyond the  $N$ th term. In more compact language :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \forall n > N.$$

### Notes

- If a sequence has a limit we say that it **converges** or **is convergent**. If not we say that it **diverges** or **is divergent**.
- If a sequence converges to  $L$ , then no matter how small a radius around  $L$  we choose, there is a point in the sequence beyond which all terms are within that radius of  $L$ . So beyond this point, all terms of the sequence are *very close together* (and very close to  $L$ ). Where that point is depends on how you interpret “very close together”.

## Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

### Example 56

- 1  $(\max\{(-1)^n, 0\})_{n=1}^{\infty} : 0, 1, 0, 1, 0, 1, \dots$   
*This sequence alternates between 0 and 1 and does not approach any limit.*
- 2 A sequence can be divergent by having terms that increase (or decrease) without limit.  
 $(2^n)_{n=1}^{\infty} : 2, 4, 8, 16, 32, 64, \dots$
- 3 A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose  $n$ th term is the  $n$ th digit after the decimal point in the decimal representation of  $\pi$ .

## Convergence is a precise concept!

**Remark:** The notion of a convergent sequence is sometimes described informally with words like “the terms get closer and closer to  $L$  as  $n$  gets larger”. It is **not true** however that the terms in a sequence that converges to a limit  $L$  must get **progressively** closer to  $L$  as  $n$  increases.

### Example 57

The sequence  $(a_n)$  is defined by

$$a_n = 0 \text{ if } n \text{ is even, } a_n = \frac{1}{n} \text{ if } n \text{ is odd.}$$

This sequence begins :

$$1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \frac{1}{9}, 0, \dots$$

It **converges to 0** although it is not true that every step takes us closer to zero.

## Examples of convergent sequences

### Example 58

Find  $\lim_{n \rightarrow \infty} \frac{n}{2n-1}$ .

**Solution:** As if calculating a limit as  $x \rightarrow \infty$  of an expression involving a continuous variable  $x$ , divide above and below by  $n$ .

$$\lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{n/n}{2n/n - 1/n} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}.$$

So the sequence  $\left(\frac{n}{2n-1}\right)$  converges to  $\frac{1}{2}$ .

## Bounded Sequences

As for subsets of  $\mathbb{R}$ , there is a concept of **boundedness** for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of  $\mathbb{R}$ , is bounded (or bounded above or bounded below). More precisely :

### Definition 59

The sequence  $(a_n)$  is **bounded above** if there exists a real number  $M$  for which  $a_n \leq M$  for all  $n \in \mathbb{N}$ .

The sequence  $(a_n)$  is **bounded below** if there exists a real number  $m$  for which  $m \leq a_n$  for all  $n \in \mathbb{N}$ .

The sequence  $(a_n)$  is **bounded** if it is bounded both above and below.

### Example 60

The sequence  $(n)$  is bounded below (for example by 0) but not above.  
The sequence  $(\sin n)$  is bounded below (for example by  $-1$ ) and above (for example by 1).

## Increasing and decreasing sequences

### Definition 62

A sequence  $(a_n)$  is called **increasing** if  $a_n \leq a_{n+1}$  for all  $n \geq 1$ .

A sequence  $(a_n)$  is called **strictly increasing** if  $a_n < a_{n+1}$  for all  $n \geq 1$ .

A sequence  $(a_n)$  is called **decreasing** if  $a_n \geq a_{n+1}$  for all  $n \geq 1$ .

A sequence  $(a_n)$  is called **strictly decreasing** if  $a_n > a_{n+1}$  for all  $n \geq 1$ .

### Definition 63

A sequence is called **monotonic** if it is either **increasing** or **decreasing**.

Similar terms: monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

**Note:** These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.

## Convergent $\implies$ Bounded

### Theorem 61

*If a sequence is convergent it must be bounded.*

**Proof** Suppose that  $(a_n)_{n=1}^{\infty}$  is a convergent sequence with limit  $L$ .

Then (by definition of convergence) there exists a natural number  $N$  such that every term of the sequence after  $a_N$  is between  $L - 1$  and  $L + 1$ .

The set consisting of the first  $N$  terms of the sequence is a finite set : it has a maximum element  $M_1$  and a minimum element  $m_1$ .

Let  $M = \max\{M_1, L + 1\}$  and let  $m = \min\{m_1, L - 1\}$ . Then  $(a_n)$  is bounded above by  $M$  and bounded below by  $m$ .

So our sequence is bounded.

## Examples

- 1 An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1, 2, 3, \dots$$

- 2 A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1, 1, -1, 1, -1, \dots$$

- 3 A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots$$

- 4 A monotonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is **bounded** and **monotonic**, it is **convergent**. This is the **Monotone Convergence Theorem**.

## The Monotone Convergence Theorem

### Theorem 64

If a sequence  $(a_n)_{n=1}^{\infty}$  is monotonic and bounded, then it is convergent.

**Proof:** Suppose that  $(a_n)$  is increasing and bounded.

Then the set  $\{a_n : n \in \mathbb{N}\}$  is a bounded subset of  $\mathbb{R}$  and by the Axiom of Completeness it has a least upper bound (or supremum)  $L$ .

We will show that the sequence  $(a_n)$  converges to  $L$ .

Choose a (very small)  $\varepsilon > 0$ . Then  $L - \varepsilon$  is not an upper bound for  $\{a_n : n \in \mathbb{N}\}$ , because  $L$  is the least upper bound.

This means there is some  $N \in \mathbb{N}$  for which  $L - \varepsilon < a_N$ . Since  $L$  is an upper bound for  $\{a_n : n \in \mathbb{N}\}$ , this means

$$L - \varepsilon < a_N \leq L$$

## Proof of the Monotone Convergence Theorem (continued)

$$L - \varepsilon < a_N \leq L$$

Since the sequence  $(a_n)$  is increasing and its terms are bounded above by  $L$ , every term after  $a_N$  is between  $a_N$  and  $L$ , and therefore between  $L - \varepsilon$  and  $L$ . These terms are all within  $\varepsilon$  of  $L$ .

Using the fact that our sequence is increasing and bounded, we have

- Identified  $L$  as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an  $\varepsilon$  we take, there is a point in our sequence beyond which all terms are within  $\varepsilon$  of  $L$ .

This is exactly what it means for the sequence to converge to  $L$ .

## An Example (from 2011 Summer Exam)

### Example 65

A sequence  $(a_n)$  of real numbers is defined by

$$a_0 = 4, \quad a_n = \frac{1}{2}(a_{n-1} - 2) \text{ for } n \geq 1.$$

- 1 Write down the first four terms of the sequence.
- 2 Show that the sequence is bounded below.
- 3 Show that the sequence is monotonically decreasing.
- 4 State why it can be deduced that the sequence is convergent, and determine its limit.

**Note:** This is an example of a sequence that is defined recursively. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the  $n$ th term although one may exist.

## Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 83.