Convergence of a series

Definition 72

For a series $\sum_{n=1}^{\infty} a_n$, and for $k \geq 1$, let

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

Thus $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$ etc.
Then s_k is called the kth partial sum of the series, and the sequence $\{s_k\}_{k=1}^{\infty}$ is called the sequence of partial sums of the series.
If the sequence of partial sums converges to a limit s, the series is said to converge and s is called its sum. In this situation we can write $\sum_{n=1}^{\infty} a_n = s$. If the sequence of partial sums diverges, the series is said to diverge.

Convergence of a geometric series

Recall Example 2 above:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

In this example, for $k \geq 0$,

$$s_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 \quad + \quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k}$$

$$\frac{1}{2} s_k = \sum_{n=1}^{k} \frac{1}{2^{n+1}} = \quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

Then

$$s_k - \frac{1}{2} s_k = \frac{1}{2} s_k = 1 - \frac{1}{2^{k+1}} \implies s_k = 2 - \frac{1}{2^k}.$$

So the sequence of partial sums has kth term $2 - \frac{1}{2^k}$. This sequence converges to 2 so the series converges to 2.

General geometric series

Consider the sequence of partial sums for the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$$

(This is a geometric series with initial term a and common ratio r.) The kth partial sum s_k is given by

$$\begin{array}{lclclclclcl} s_k & = & \sum_{n=0}^{k} ar^n & = & a + & ar + & \ldots & + & ar^k & & \\ rs_k & = & \sum_{n=0}^{k} ar^{n+1} & = & & ar + & ar^2 + & \ldots + & ar^k & + & ar^{k+1} \end{array}$$

Then $(1-r)s_k = a - ar^{k+1} \Longrightarrow s_k = \dfrac{a(1-r^{k+1})}{1-r}$. If $|r| < 1$, then $r^{k+1} \to 0$ as $k \to \infty$, and the sequence of partial sums (hence the series) converges to $\dfrac{a}{1-r}$. If $|r| \geq 1$ the series is divergent.

Theorem 73

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Proof: Think of $\frac{1}{n}$ as the area of a rectangle of height $\frac{1}{n}$ and width 1, sitting on the interval $[n, n+1]$ on the x-axis. So the $\frac{1}{1}$ corresponds to a square of area 1 sitting on the interval $[1, 2]$, the term $\frac{1}{2}$ corresponds to a rectangle of area $\frac{1}{2}$ sitting on the interval $[2, 3]$ and so on.
The total area accounted for by these triangles is the sum of the harmonic series, and this exceeds the area accounted for by the improper integral

$$\int_1^{\infty} \frac{1}{x}\,dx.$$

From Section 1.5 we know that this area is infinite.

A necessary condition for convergence

Note: A necessary condition for the series

$$\sum_{n=1}^{\infty} a_n$$

to converge is that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0; i.e. that $a_n \to 0$ as $n \to \infty$. If this does *not* happen, then the sequence of partial sums has no possibility of converging.

The example of the harmonic series shows that the condition $a_n \to 0$ as $n \to \infty$ is not sufficient to guarantee that the series $\sum_{n=1}^{\infty}$ will converge.

Learning outcomes for Section 3.3

After studying this section you should be able to

- explain what an infinite series is and what it means for an infinite series to converge;
- Give examples of convergent and divergent series;
- show that the harmonic series is divergent;
- Use the "sigma" notation for sums.

Section 3.4: Introduction to power series

Definition 74

A power series in the variable x resembles a polynomial, except that it may contain infinitely many positive powers of x. It is an expression of the type

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots,$$

where each a_i is a number.

Example 75

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is a power series.

Question: Can we think of a power series as a function of x?

Power Series as Functions

Define a "function" by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$$

- If we try to evaluate this function at $x = 2$, we get a series of real numbers.

  $$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \ldots$$

  This series is divergent, so our power series does not define a function that can be evaluated at 2.
- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

  $$f(0) = 1 + 0 + 0^2 + \cdots = 1.$$

  So it does make sense to "evaluate" this function at $x = 0$.

- If we try evaluating at $x = \frac{1}{2}$, we get

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

  This is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\dfrac{a}{1-r}$. In this case

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

  and we have $f(\frac{1}{2}) = 2$.

So we can evaluate our function at $x = \frac{1}{2}$.

A geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1, 1)$), we find that $f(x)$ is a convergent geometric series, converging to $\dfrac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1, 1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

The interval $(-1, 1)$ is called the interval of convergence of the power series, and 1 is the radius of convergence. We say that the power series representation of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1, 1)$.

Which functions have power series representations?

Remark: The power series representation is not particularly useful if you want to calculate $\frac{1}{1-x}$ for some particular value of x, because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $\sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are not easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it?

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of $\mathbb{R}$? If a function could be represented by a power series, how would we calculate the coefficients in this series?