

## Section 3.3: Introduction to Infinite Series

### Definition 66

A **series** or **infinite series** is the sum of all the terms in a sequence.

### Example 67 (Examples of infinite series)

1  $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

2 A **geometric series**

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Every term in this series is obtained from the previous one by multiplying by the **common ratio**  $\frac{1}{2}$ . This is what **geometric** means.

## Notes

- 1 For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.
- 2 A **series** is not the same thing as a **sequence** - don't confuse these terms! A **sequence** is a list of numbers. A **series** is an infinite sum.
- 3 The "sigma" notation for sums: **sigma** (lower case  $\sigma$ , upper case  $\Sigma$ ) is a letter from the Greek alphabet, the upper case  $\Sigma$  is used to denote sums. The notation  $\sum_{n=i}^j a_n$  means:  
 $i$  and  $j$  are integers and  $i \leq j$ . For each  $n$  from  $i$  to  $j$  the number  $a_n$  is defined; the expression above means the sum of the numbers  $a_n$  where  $n$  runs through all the values from  $i$  to  $j$ , i.e.

$$\sum_{n=i}^j a_n = a_i + a_{i+1} + a_{i+2} + \dots + a_{j-1} + a_j.$$

For infinite sums we can have  $-\infty$  and/or  $\infty$  (instead of fixed integers  $i$  and  $j$ ) as subscripts and superscripts for the summation.

## Examples of Series (continued)

### Example 68

3. The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. An **alternating series**

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

## Sequences of partial sums

In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1  $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, 1 + 2 + 3 + 4 + 5, ... **1, 3, 6, 10, 15, ...**

Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as  $n$  increases) - we can't associate a numerical value with this infinite sum.

## Examples (continued)

### 2. A geometric series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{2^2}, 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \dots \quad 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32} \dots$$

In this example the terms that are being added on at each step ( $\frac{1}{2^n}$ ) are getting smaller and smaller as  $n$  increases, and the numbers in the list appear to be converging to 2.

### 3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \quad 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$$

It is harder to see what is going on here.

## Notes

### 4. An alternating series

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

$$1, 1 - 1, 1 - 1 + 1, 1 - 1 + 1 - 1, 1 - 1 + 1 - 1 + 1 \dots \quad 1, 0, 1, 0, 1, \dots$$

The terms being “added on” at each step are alternating between 1 and  $-1$ , and as we proceed with the summation the “running total” alternates between 0 and 1. There is no numerical value that we can associate with the infinite sum  $\sum_{n=0}^{\infty} (-1)^n$ .

**Note:** The series in 2. above **converges** to 2, the series in 1. and 4. are both **divergent** and it is not obvious yet but the series in 3. is **divergent** as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. We know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.

## Convergence of a series

### Definition 69

For a series  $\sum_{n=1}^{\infty} a_n$ , and for  $k \geq 1$ , let

$$s_k = \sum_{n=1}^k a_n = a_1 + a_2 + a_3 + \dots + a_k.$$

Thus  $s_1 = a_1$ ,  $s_2 = a_1 + a_2$ ,  $s_3 = a_1 + a_2 + a_3$  etc.

Then  $s_k$  is called the  **$k$ th partial sum** of the series, and the sequence  $\{s_k\}_{k=1}^{\infty}$  is called the **sequence of partial sums** of the series.

If the sequence of partial sums converges to a limit  $s$ , the series is said to **converge** and  $s$  is called its sum. In this situation we can write  $\sum_{n=1}^{\infty} a_n = s$ . If the sequence of partial sums **diverges**, the series is said to **diverge**.

## Convergence of a geometric series

Recall Example 2 above:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

In this example, for  $k \geq 0$ ,

$$s_k = \sum_{n=0}^k \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k}$$

$$\frac{1}{2} s_k = \sum_{n=1}^k \frac{1}{2^{n+1}} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

Then

$$s_k - \frac{1}{2} s_k = \frac{1}{2} s_k = 1 - \frac{1}{2^{k+1}} \implies s_k = 2 - \frac{1}{2^k}.$$

So the sequence of partial sums has  $k$ th term  $2 - \frac{1}{2^k}$ . This sequence converges to 2 so the series converges to 2.

## General geometric series

Consider the sequence of partial sums for the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

(This is a **geometric series** with initial term  $a$  and **common ratio**  $r$ .) The  $k$ th partial sum  $s_k$  is given by

$$\begin{aligned} s_k &= \sum_{n=0}^k ar^n = a + ar + \dots + ar^k \\ rs_k &= \sum_{n=0}^k ar^{n+1} = ar + ar^2 + \dots + ar^k + ar^{k+1} \end{aligned}$$

Then  $(1-r)s_k = a - ar^{k+1} \Rightarrow s_k = \frac{a(1-r^{k+1})}{1-r}$ . If  $|r| < 1$ , then  $r^{k+1} \rightarrow 0$  as  $k \rightarrow \infty$ , and the sequence of partial sums (hence the series) converges to  $\frac{a}{1-r}$ . If  $|r| \geq 1$  the series is divergent.

## A necessary condition for convergence

**Note:** A necessary condition for the series

$$\sum_{n=1}^{\infty} a_n$$

to **converge** is that the **sequence**  $\{a_n\}_{n=1}^{\infty}$  converges to 0; i.e. that  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . If this does *not* happen, then the sequence of partial sums has no possibility of converging.

The example of the harmonic series shows that the condition  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  is not **sufficient** to guarantee that the series  $\sum_{n=1}^{\infty} a_n$  will converge.

## The harmonic series is divergent

### Theorem 70

The harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is divergent.

**Proof:** Think of  $\frac{1}{n}$  as the area of a rectangle of height  $\frac{1}{n}$  and width 1, sitting on the interval  $[n, n+1]$  on the  $x$ -axis. So the  $\frac{1}{1}$  corresponds to a square of area 1 sitting on the interval  $[1, 2]$ , the term  $\frac{1}{2}$  corresponds to a rectangle of area  $\frac{1}{2}$  sitting on the interval  $[2, 3]$  and so on.

The total area accounted for by these rectangles is the sum of the harmonic series, and this exceeds the area accounted for by the improper integral

$$\int_1^{\infty} \frac{1}{x} dx.$$

From Section 1.5 we know that this area is infinite.

## Learning outcomes for Section 3.3

After studying this section you should be able to

- explain what an infinite series is and what it means for an infinite series to converge;
- Give examples of convergent and divergent series;
- show that the harmonic series is divergent;
- Use the “sigma” notation for sums.