A power series in the variable \(x\) resembles a polynomial, except that it may contain infinitely many positive powers of \(x\). It is an expression of the type
\[
\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots
\]
where each \(a_i\) is a number.

\[\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots\]
is a power series.

**Question:** Can we think of a power series as a function of \(x\)?

Define a “function” by
\[
f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots
\]

- If we try to evaluate this function at \(x = 2\), we get a series of real numbers.
  \[
f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \ldots
\]
  This series is divergent, so our power series does not define a function that can be evaluated at 2.

- If we try evaluating at 0 (and allow that the first term \(x^0\) of the power series is interpreted as 1 for all values of \(x\)), we get
  \[
f(0) = 1 + 0 + 0^2 + \cdots = 1.
\]
  So it does make sense to “evaluate” this function at \(x = 0\).

If we try evaluating at \(x = \frac{1}{2}\), we get
\[
f \left( \frac{1}{2} \right) = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = 1 + \frac{1}{2} + \left( \frac{1}{2} \right)^2 + \ldots
\]
This is a geometric series with first term \(a = 1\) and common ratio \(r = \frac{1}{2}\). We know that if \(|r| < 1\), such a series converges to the number \(\frac{a}{1-r}\). In this case
\[
\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,
\]
and we have \(f \left( \frac{1}{2} \right) = 2\).

So we can evaluate our function at \(x = \frac{1}{2}\).
Maclaurin (or Taylor) series

Which functions have power series representations?

A geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. In general for any value of \( x \) whose absolute value is less than 1 (i.e. any \( x \) in the interval \((-1, 1)\)), we find that \( f(x) \) is a convergent geometric series, converging to \( \frac{1}{1-x} \).

Conclusion: For values of \( x \) in the interval \((-1, 1)\) (i.e. \( |x| < 1 \)), the function \( f(x) = \frac{1}{1-x} \) coincides with the power series \( \sum_{n=0}^{\infty} x^n \). The interval \((-1, 1)\) is called the interval of convergence of the power series, and 1 is the radius of convergence. We say that the power series representation of the function \( f(x) = \frac{1}{1-x} \) is \( \sum_{n=0}^{\infty} x^n \), for values of \( x \) in the interval \((-1, 1)\).

Maclaurin (or Taylor) series

Maclaurin (or Taylor) series

Suppose that \( f(x) \) is an infinitely differentiable function (this means that all the derivatives of \( f \) are themselves differentiable), and suppose that \( f \) is represented by the power series

\[
f(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

We can work out appropriate values for the coefficients \( c_n \) as follows.

- Put \( x = 0 \). Then \( f(0) = c_0 + \sum_{n=1}^{\infty} c_n(0)^n \Rightarrow f(0) = c_0 \).

The constant term in the power series is the value of \( f \) at 0.

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The constant term in the power series is the value of \( f \) at 0.

- To calculate \( c_1 \), look at the value of the first derivative of \( f \) at 0, and differentiate the power series term by term. We expect

\[
f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} nc_n x^{n-1}.
\]

Then we should have \( f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \cdots = c_1 \). Thus

\[c_1 = f'(0)\]
\[ f(x) = \sum_{n=0}^{\infty} c_n x^n \]

For \( c_2 \), look at the second derivative of \( f \). We expect
\[ f''(x) = 2(1)c_2 + 3(2)c_3 x + 4(3)c_4 x^2 + 5(4)c_5 x^3 + \ldots \]
Putting \( x = 0 \) gives \( f''(0) = 2(1)c_2 \) or
\[ c_2 = \frac{f''(0)}{2(1)}. \]

For \( c_3 \), look at the third derivative \( f'''(x) \). We have
\[ f'''(x) = 3(2)(1)c_3 + 4(3)(2)c_4 x + 5(4)(3)c_5 x^2 + \ldots \]
Setting \( x = 0 \) gives \( f'''(0) = 3(2)(1)c_3 \) or
\[ c_3 = \frac{f'''(0)}{3(2)(1)}. \]

**Coefficients of the Maclaurin Series**

Continuing this process, we obtain the following general formula for \( c_n \):
\[ c_n = \frac{1}{n!} f^{(n)}(0). \]

**Definition 72**

For a positive integer \( n \), the number \( n \) factorial, denoted \( n! \) is defined by
\[ n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1. \]

The number 0! (zero factorial) is defined to be 1.

Write \( f(x) = \sin x \), and write \( \sum_{n=0}^{\infty} c_n x^n \) for the Maclaurin series of \( \sin x \). Then
\[ f(0) = \sin 0 = 0 \implies c_0 = 0. \]
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- \( f(0) = \sin 0 = 0 \implies c_0 = 0 \)
- \( f'(0) = \cos 0 = 1 \implies c_1 = 1 \)

Write \( f(x) = \sin x \), and write \( \sum_{n=0}^{\infty} c_n x^n \) for the Maclaurin series of \( \sin x \). Then

- \( f(0) = \sin 0 = 0 \implies c_0 = 0 \)
- \( f'(0) = \cos 0 = 1 \implies c_1 = 1 \)
- \( f''(0) = -\sin 0 = 0 \implies c_2 = \frac{2}{2!} = 0 \)
- \( f^{(3)}(0) = -\cos 0 = -1 \implies c_3 = \frac{-1}{3!} = -\frac{1}{6} \)
Power series representation of $\sin x$

This pattern continues:
- If $k$ is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If $k$ is odd and $k \equiv 1 \pmod{4}$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + ...$$

Note that this series only involves odd powers of $x$ - this is not surprising because $\sin$ is an odd function; it satisfies $\sin(-x) = -\sin x$.

Power series representations of $\sin x$ and $\cos x$

Theorem 73

For every real number $x$, the above series converges to $\sin x$.

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number $x$.

Exercise 74

Show that the Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$  

(Note that this can be obtained by differentiating term-by-term the series for $\sin x$, as we would expect since $\frac{d}{dx}(\sin x) = \cos x$.)
After studying this section you should be able to

- State the meaning of the term power series,
- Explain the concept of the radius of convergence of a power series,
- Calculate the coefficients in (an initial segment of) the Maclaurin series representation of a given function.