Section 3.4: Introduction to power series

Definition 71

A power series in the variable x resembles a polynomial, except that it may contain infinitely many positive powers of x. It is an expression of the type

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots,$$

where each a_i is a number.

Example 72

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

is a power series.

Question: Can we think of a power series as a function of x?

Power Series as Functions

Define a "function" by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

- If we try to evaluate this function at $x = 2$, we get a series of real numbers.

  $$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \dots$$

  This series is divergent, so our power series does not define a function that can be evaluated at 2.
- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

  $$f(0) = 1 + 0 + 0^2 + \dots = 1.$$

  So it does make sense to "evaluate" this function at $x = 0$.

$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$

- If we try evaluating at $x = \frac{1}{2}$, we get

  $$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

  This is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\frac{a}{1-r}$. In this case

  $$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

  and we have $f(\frac{1}{2}) = 2$.

So we can evaluate our function at $x = \frac{1}{2}$.

$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$

A geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1, 1)$), we find that $f(x)$ is a convergent geometric series, converging to $\frac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1, 1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1.$$

The interval $(-1, 1)$ is called the interval of convergence of the power series, and 1 is the radius of convergence. We say that the power series representation of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1, 1)$.

Which functions have power series representations?

Remark: The power series representation is not particularly useful if you want to calculate $\frac{1}{1-x}$ for some particular value of x, because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $\sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are not easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it?

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of $\mathbb{R}$? If a function could be represented by a power series, how would we calculate the coefficients in this series?

Maclaurin (or Taylor) series

Suppose that $f(x)$ is an infinitely differentiable function (this means that all the deriviatives of f are themselves differentiable), and suppose that f is represented by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can work out appropriate values for the coefficients c_n as follows.

- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n (0)^n \implies f(0) = c_0$.
  The constant term in the power series is the value of f at 0.
- To calculate c_1, look at the value of the first derivative of f at 0, and differentiate the power series term by term. We expect

  $$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

  Then we should have $f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \cdots = c_1$. Thus

  $$c_1 = f'(0).$$

$f(x) = \sum_{n=0}^{\infty} c_n x^n$

- For c_2, look at the second derivative of f. We expect

  $$f''(x) = 2(1)c_2 + 3(2)c_3 x + 4(3)c_4 x^2 + 5(4)c_5 x^3 + \dots$$

  Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or

  $$c_2 = \frac{f''(0)}{2(1)}.$$

- For c_3, look at the third derivative $f^{(3)}(x)$. We have

  $$f^{(3)}(x) = 3(2)(1)c_3 + 4(3)(2)c_4 x + 5(4)(3)c_5 x^2 + \dots$$

  Setting $x = 0$ gives $f^{(3)}(0) = 3(2)(1)c_3$ or

  $$c_3 = \frac{f^{(3)}(0)}{3(2)(1)}$$

Coefficients of the Maclaurin Series

Continuing this process, we obtain the following general formula for c_n:

$$c_n = \frac{1}{n!} f^{(n)}(0).$$

Definition 73

For a positive integer n, the number n factorial, denoted $n!$ is defined by

$$n! = n \times (n-1) \times (n-2) \times \dots 3 \times 2 \times 1.$$

The number 0! (zero factorial) is defined to be 1.

Power series representation of $\sin x$

Write $f(x) = \sin x$, and write $\sum_{n=0}^{\infty} c_n x^n$ for the Maclaurin series of $\sin x$. Then

- $f(0) = \sin 0 = 0 \implies c_0 = 0$
- $f'(0) = \cos 0 = 1 \implies c_1 = 1$
- $f''(0) = -\sin 0 = 0 \implies c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \implies c_3 = \frac{-1}{3!} = -\frac{1}{6}$
- $f^{(4)}(0) = \sin 0 = 0 \implies c_4 = \frac{0}{4!} = 0$

Power series representation of $\sin x$

This pattern continues :

- If k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.
- If k is odd and $k \equiv 1 \mod 4$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.
- If k is odd and $k \equiv 3 \mod 4$ then $f^{(k)}(0) = -\cos 0 = -1$ and $c_k = -\frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Note that this series only involves odd powers of x - this is not surprising because sin is an odd function; it satisfies $\sin(-x) = -\sin x$.

Power series representations of $\sin x$ and $\cos x$

Theorem 74

For every real number x, the above series converges to $\sin x$.

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number x.

Exercise 75

Show that the Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k!)} x^{2k}.$$

(Note that this can be obtained by differentiating term-by-term the series for $\sin x$, as we would expect since $\frac{d}{dx}(\sin x) = \cos x$.)

Learning outcomes for Section 3.4

After studying this section you should be able to

- State the meaning of the term *power series*,
- Explain the concept of the *radius of convergence* of a power series,
- Calculate the coefficients in (an initial segment of) the Maclaurin series representation of a given function.