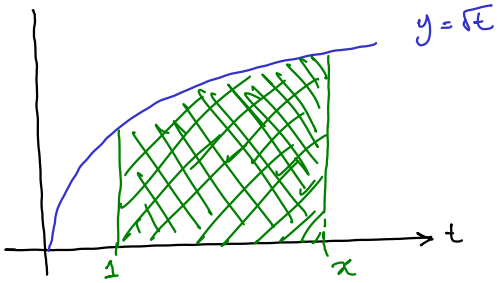
SAMPLE EXAM QUESTIONS FROM LECTURE 24

SOLUTIONS

1. (a) (i) $A(x) = \int_1^x \sqrt{t}\, dt$

$A(x)$ is the area shaded in green.



(ii) $A'(x) = \sqrt{x}$, for $x > 1$

(iii) $A'(4) = 2$

(b) (i) $\int_1^2 x e^{3x}\, dx$

Use integration by parts with $u = x$, $v' = e^{3x}$

$u' = 1$ $\quad v = \frac{1}{3} e^{3x}$

$$\int_1^2 uv'\, dx = uv\Big|_1^2 - \int_1^2 u'v\, dx = \frac{1}{3} e^{3x}\Big|_1^2 - \frac{1}{3}\int_1^2 e^{3x}\, dx$$

$$= \frac{1}{3}e^6 - \frac{1}{3}e^3 - \frac{1}{9}e^{3x}\Big|_1^2$$

$$= \frac{1}{3}e^6 - \frac{1}{3}e^3 - \frac{1}{9}e^6 + \frac{1}{9}e^3 = \boxed{\frac{2}{9}\left(e^6 - e^3\right)}$$

(ii) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\, dx$.

Write $u = \sqrt{x}$. Then $du/dx = \frac{1}{2\sqrt{x}} \Rightarrow \frac{1}{\sqrt{x}}\, dx = 2\, du$

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = 2\int \cos u\, du = 2\sin u + C = \boxed{2\sin(\sqrt{x}) + C}$$

(iii) $\int \frac{x^3 + 3x + 3}{x+1}\, dx$

$$x^3 + 3x + 3 = (x+1)(x^2 - x + 4) - 1 \Rightarrow \frac{x^3+3x+3}{x+1} = x^2 - x + 4 - \frac{1}{x+1}$$

$$\int \frac{x^3+3x+3}{x+1}\, dx = \int \left(x^2 - x + 4 - \frac{1}{x+1}\right) dx$$

$$= \boxed{\frac{x^3}{3} - \frac{x^2}{2} + 4x - \ln|x+1| + C}$$

2. (a) (i) Every unbounded subset of $\mathbb{R}$ is infinite.

TRUE Every finite set must be bounded, since it must have a greatest and a least element.

(ii) Every infinite subset of $\mathbb{R}$ is unbounded

FALSE. For example $(0,1)$ is infinite but bounded.

(iii) If a subset of $\mathbb{Q}$ is bounded above, then it has a maximum element.

FALSE For example $(0,1) \cap \mathbb{Q}$ is bounded above (by 1) but has no maximum element.

(b) A set is countable if it can be put in bijective correspondence with the set $\mathbb{N}$ of natural numbers. A bijective correspondence between $\mathbb{N}$ and $\mathbb{Z}$ is given by:

$\mathbb{Z}$	$\mathbb{N}$
0	1
1	2
-1	3
2	4
-2	5
3	6

$f: \mathbb{Z} \to \mathbb{N}$ $\quad f(0) = 1$, $f(n) = 2n$ if $n > 0$, $f(n) = -2n+1$ for $n < 0$.

(c) $S = \left\{ \frac{2n+1}{n^2} : n \in \mathbb{Z}, n \geq 1 \right\}$

(i) Four elements of S: $3, \ 5/4, \ 7/9, \ 9/16$

(ii) S is bounded below since every element of S is positive

To see that S is bounded above, note that elements of S have the form $\frac{2}{n} + \frac{1}{n^2}$. When n is a positive integer, $2/n$ and $1/n^2$ are bounded above by 2 and 1 respectively. So $\frac{2n+1}{n^2}$ is bounded above by 3.

(iii) 3 is the maximum element of S, and hence also the supremum. S does not have a minimum element. Its infimum is 1. While all elements of S are positive, by choosing n great enough we can arrange for $\frac{2}{n} + \frac{1}{n^2}$ to be within ε of 0, no matter how small the positive real number ε is.

Q3 (a) (i) A convergent sequence of real numbers:

$\left(\frac{1}{n}\right)_{n\geq 1}$: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ converges to 0.

(ii) A sequence of real numbers that is bounded and divergent

$\left((-1)^n\right)_{n\geq 1}$: $-1, 1, -1, 1, -1, 1, \ldots$

(iii) A sequence that is convergent and not monotonic:

$\left(\frac{(-1)^n}{n}\right)_{n\geq 1}$: $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$

(b) $a_0 = 4$, $a_n = \sqrt{a_{n-1}^2 - 2a_{n-1} + 4}$, for $n \geq 1$.

(i) First 4 terms:

$a_0 = 4$, $a_1 = \sqrt{12}$, $a_2 = \sqrt{16 - 2\sqrt{12}}$, $a_3 = \sqrt{20 - 2\sqrt{12} - 2\sqrt{16 - 2\sqrt{12}}}$

(ii) $a_0 > 2$ by definition.

Suppose for some $n \geq 1$ that $a_{n-1} > 2$.

Then $a_n = \sqrt{a_{n-1}(a_{n-1} - 2) + 4}$

($a_{n-1} > 2$, $a_{n-1} - 2 > 0$)

Since $a_n = \sqrt{t+4}$ where $t > 0$, it follows that $a_n > 2$.

So $a_n > 2$ for all n, by induction

(iii) We need to show $a_n \leq a_{n-1}$ for all $n \geq 1$.

$a_n = \sqrt{a_{n-1}^2 - 2(a_{n-1} - 2)}$

Since $a_{n-1} - 2$ is positive from (ii) above,

$a_{n-1}^2 - 2(a_{n-1} - 2) < a_{n-1}^2$

Hence $a_n < \sqrt{a_{n-1}^2} = a_{n-1}$, and (a_n) is decreasing

(iv) The sequence is convergent by the Monotone Convergence Theorem, since it is decreasing and bounded below.

The limit L must satisfy

$L = \sqrt{L^2 - 2L + 4} \Rightarrow L^2 = L^2 - 2L + 4 \Rightarrow \boxed{L = 2}$

(c) Write $f(x) = \frac{1}{1-x}$ $\quad f'(x) = \frac{1}{(1-x)^2}$ $\quad f''(x) = \frac{2}{(1-x)^3}$ $\quad f'''(x) = \frac{6}{(1-x)^4}$

$f(0) = 1$ $\quad f'(0) = 1$ $\quad f''(0) = 2$ $\quad f'''(0) = 6$

Maclaurin Series: $f(0) + f'(0)x + f''(0)/2!\, x^2 + f'''(0)/3!\, x^3 + \ldots$

$1 + x + x^2 + x^3 + \ldots$