Chapter 1

Integral Calculus

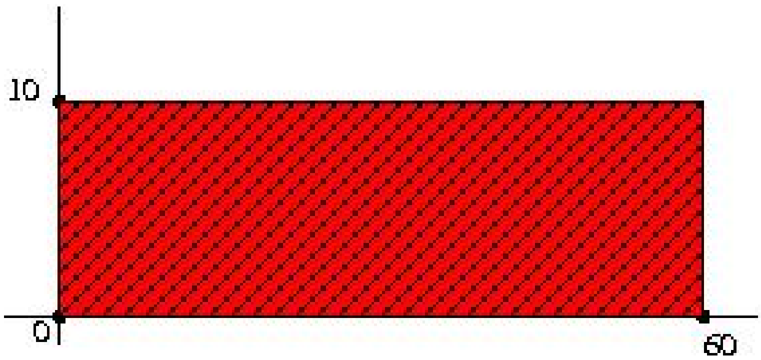
1.1 Areas under curves - introduction and examples

Problem 1.1.1. *A car travels in a straight line for one minute, at a constant speed of 10m/s. How far has the car travelled in this minute?*

SOLUTION: The car is travelling for 60 seconds, and covering 10 metres in each second, so in total it covers $60 \times 10 = 600$ metres.
That wasn't very hard.

An easy example like this one can be a starting point for studying more complicated problems. What makes this example easy is that the car's speed is not changing so all we have to do is multiply the distance covered in one second by the number of seconds. Note that we can interpret the answer graphically as follows.

Suppose we draw a graph of the car's speed against time, where the x-axis is labelled in seconds and the y-axis in m/s. The graph is just the horizontal line $y = 10$ of course.



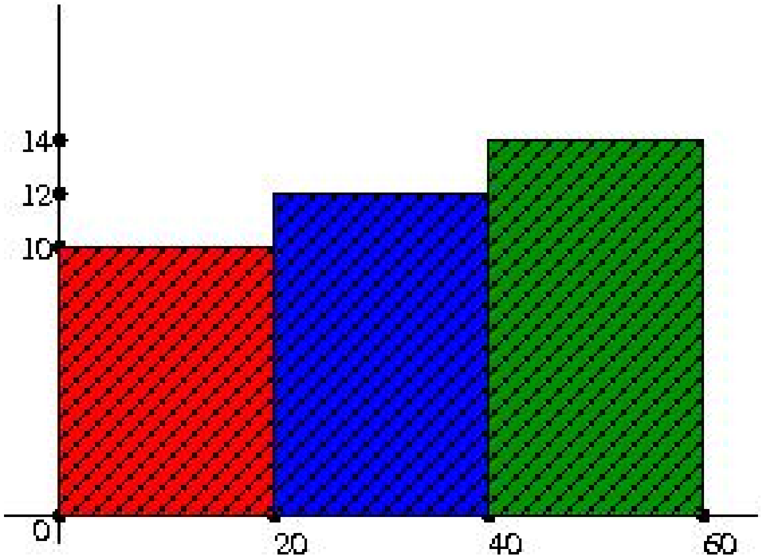
We can label the time when we start observing the car's motion as $t = 0$ and the time when we stop as $t = 60$. Note then that the total distance travelled – 600m – is the area enclosed under the graph, between the x-axis, the horizontal line $y = 10$, and the vertical lines $x = 0$ (or time $t = 0$) and $x = 60$ marking the beginning and end of the period of observation. This is no coincidence; if we divide this rectangular region into vertical strips of width 1, one for each second, what we get are 60 vertical strips of width 1 and height 10, each accounting for 10 units of area, and each accounting for 10 metres of travel.
The next problem is a slightly harder example of the same type.

Problem 1.1.2. *Again the car travels in one direction for one minute. This time it travels at 10m/s for the first 20 seconds, at 12m/s for the next 20 seconds, and at 14m/s for the last 20 seconds.*
What is the total distance travelled?

SOLUTION: This is not much harder really (although it may be a physically unrealistic problem - why?). This time, the car covers

- $20 \times 10 = 200$ metres in the first 20 seconds
- $20 \times 12 = 240$ metres in the next 20 seconds and
- $20 \times 14 = 280$ metres in the last 20 seconds,

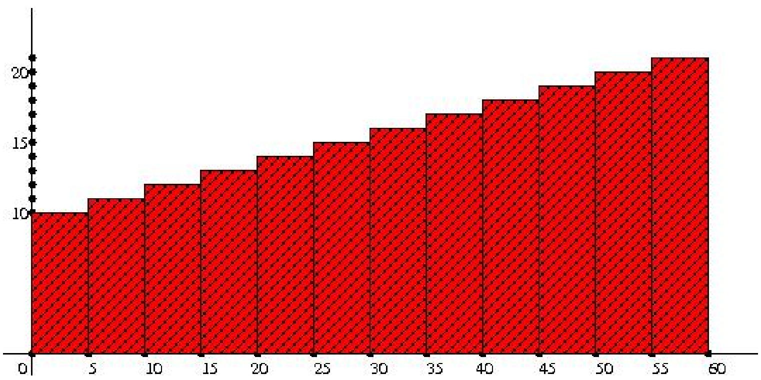
so the total distance is 720 metres.



Once again the total distance travelled is the area of the region enclosed between the lines $x = 0$, $x = 60$, the y-axis and the graph showing speed against time. The region whose area represents the distance travelled is the union of three rectangles, all of width 20, and of heights 10, 12 and 14.

Problem 1.1.3. *Same set up, but this time the car's speed is* $10\mathrm{m/s}$ *for the first 5 seconds,* $11\mathrm{m/s}$ *for the next 5, and so on, increasing by* $1\mathrm{m/s}$ *every five seconds so that the speed is 21* $\mathrm{m/s}$ *for the last five seconds. Again the problem is to calculate the total distance travelled in metres.*

The answer is left as an exercise, but this time the distance is the area indicated below.

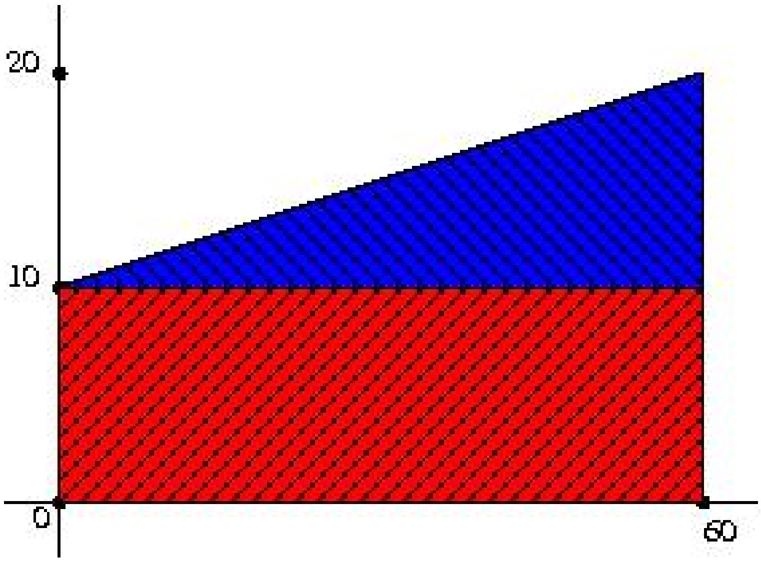


Problem 1.1.4. *Again our car is travelling in one direction for one minute, but this time its speed increases at a constant rate from* $10\mathrm{m/s}$ *at the start of the minute, to* $20\mathrm{m/s}$ *at the end. What is the distance travelled?*

NOTE: This is a more realistic problem, in which the speed is increasing at a constant rate. This constant acceleration would apply for example in the case of an object falling freely under gravity.

SOLUTION: This is a different problem from the others. Because the speed is varying *all the time* this problem cannot be solved by just multiplying the speed by the time or by a combination of such steps as in Problems 1.1.1 and 1.1.2.

The following picture shows the graph of the speed against time.



If the total distance travelled is represented in this example, as in the others, by the area under the speed graph between $t = 0$ and $t = 60$, then we can observe that it's the area of a region consisting of a rectangle of width 60 and height 10, and a triangle of width 60 and perpendicular height 10. Thus the total distance travelled is given by

$$(60 \times 10) + \frac{1}{2}(60 \times 10) = 900\mathrm{m}.$$

QUESTION: Should we believe this answer? Just because the distance is given by the area under the graph when the speed is constant, how do we know the same applies in cases where the speed is varying continuously? Here is an argument that might justify this claim.

In Problem 1.1.4, the speed increases steadily from 10m/s to 20m/S over the 60 seconds. We want to calculate the distance travelled.

We can approximate this distance as follows.

- Suppose we divide the one minute into 30 two-second intervals.
- At the start of the first two-second interval, the car is travelling at 10m/s. We make the *simplifying assumption* that the car travels at 10m/s throughout the first two seconds, thereby covering 20m in the first two seconds. Note that this actually underestimates the true distance travelled in the first second, because in fact the speed is increasing from 20m/s during these two seconds.
- At the start of the second two-second interval, the car has completed one-thirtieth of its acceleration from 10m/s to 20m/s, so its speed is

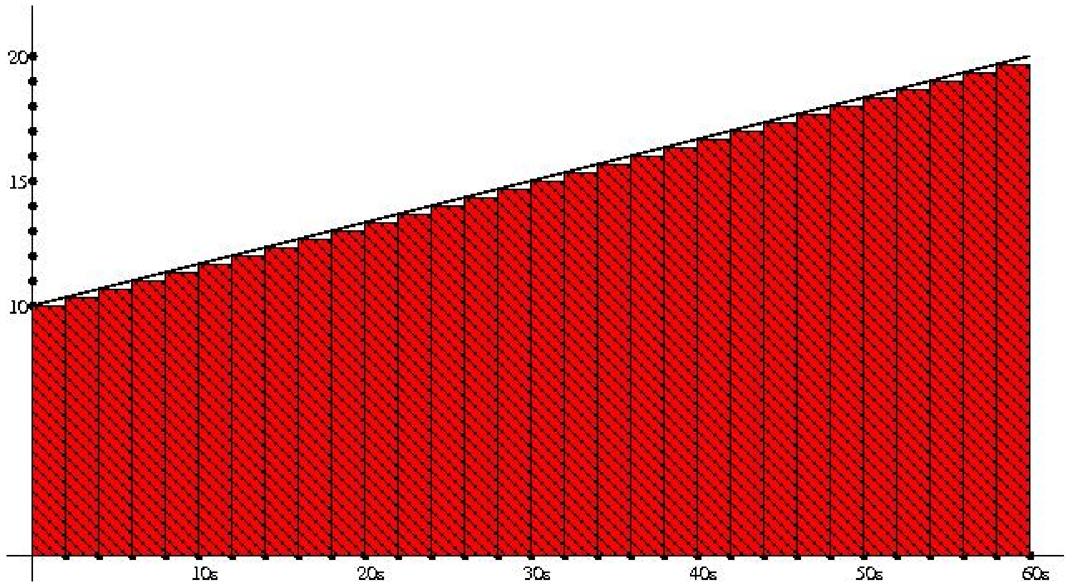
$$10 + \frac{10}{30} = 10\frac{1}{3}\mathrm{m/s}.$$

If we make the *simplifying assumption* that the speed remains constant at $10\frac{1}{3}\mathrm{m/s}$ throughout the second two-second interval, we estimate that the car travels $20\frac{2}{3}\mathrm{m}$ during the second two-second interval. This underestimates the true distance beacuse the car is actually accelerating *from* $10\frac{1}{3}\mathrm{m/s}$ during these two seconds.

- If we proceed in this manner we would estimate that the car travels
  - $20\mathrm{m}$ in the first two seconds;
  - $20\frac{2}{3}\mathrm{m}$ in the next two seconds;
  - $21\frac{1}{3}\mathrm{m}$ in the next two seconds, and so on;
  - $\ldots 39\frac{1}{3}\mathrm{m}$ in the 30th two-second interval.

This would give us a total of $890\mathrm{m}$ as the estimate for distance travelled, but that's not really the point of this discussion.

The distance that we estimate using the assumption that the speed remains constant for each of the 30 two-second intervals, is indicated by the area in red in the diagram below, where the black line is the true speed graph. Note that the red area includes all the area under the speed graph, except for 30 small triangles of base length 2 and height $\frac{1}{3}$.



Suppose now that we refine the estimate by dividing our minute of time into 60 one-second intervals and assuming the the speed remains constant for each of these, instead of into 30 two-second intervals.

If do this we will estimate that the car travels

- $10\mathrm{m}$ in the first seconds;
- $10\frac{1}{6}\mathrm{m}$ in the next seconds;
- $10\frac{2}{6}\mathrm{m}$ in the next second, and so on;
- $\ldots 19\frac{5}{6}\mathrm{m}$ in the 60th one-second interval.

This would give us a total of $895\mathrm{m}$ as the estimate for distance travelled. What is the corresponding picture? Draw it, or at least part of it, as an exercise.

Note that this *still underestimates* the distance travelled in each second, because it assumes that the speed remains constant at its starting point for the duration of each second, whereas in reality

it increases. But this estimate is closer to the true answer than the last one, because this estimate takes into account speed increases every second, instead of every two seconds.

The corresponding "area" picture has sixty rectangles of width 1 instead of thirty of width 2, and it includes all the area under the speed graph, except for *sixty triangles of base length 1 and height* $\frac{1}{6}$.

If we used the same strategy but dividing our minute into 120 half-second intervals, we would expect to get a better estimate again. As the number of intervals increases and their width decreases, the red rectangles in the picture come closer and closer to filling *all* the area under the speed graph. The true distance travelled is the limit of these improving estimates, as the length of the subintervals approaches zero. This is *exactly* the area under the speed graph, between $x = 0$ and $x = 60$.

So we can now assert more confidently that the answer to Problem 1.1.4 is 900 metres.

Note for independent study: With some careful attention you should be able check that if you split the one minute into n subintervals each of length $\frac{60}{n}$ and estimate the distance travelled as above, your answer will be $900 - \frac{300}{n}$. The limit of this expression as $n \to \infty$ is 900.

Problem 1.1.5. *Again our car is travelling in one direction for one minute, but this time its speed* v *increases from* 10m/s *to* 20m/s *over the minute, according to the formula*

$$v(t) = 20 - \frac{1}{360}(60 - t)^2,$$

where t *is measured in seconds, and* $t = 0$ *at the start of the minute.*
What is the distance travelled?

NOTE: The formula means that after t seconds have passed, the speed of the car in m/s is $20 - \frac{1}{360}(60 - t)^2$. So for example after 30 seconds the car is travelling at a speed of

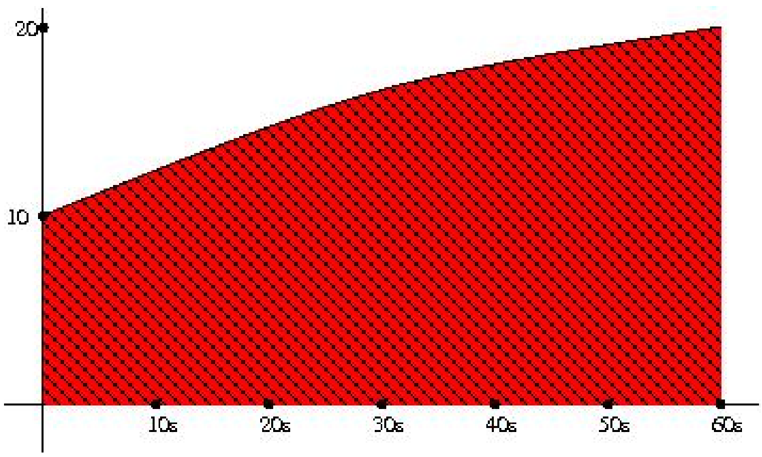
$$20 - \frac{1}{360}(60 - 30)^2 = 20 - \frac{1}{360}900 = 17.5\text{m/s}.$$

Note that $60 - t$ is decreasing as t increases from 0 to 60, so $(60 - t)^2$ is decreasing also. Thus the expression

$$20 - \frac{1}{360}(60 - t)^2$$

is *increasing* as t goes from 0 to 60. So the car is accelerating throughout the minute.

Below is the graph of the speed (in m/s) against time (in s), with the area below it (between $t = 0$ and $t = 60$) coloured red.



The argument above works in exactly the same way for this example, to persuade us that the distance travelled should be given by the area under the speed graph, between $t = 0$ and $t = 60$. This is the area that is coloured red in the picture above.

PROBLEM! The upper boundary of this area is a part of a parabola not a line segment. The region is not a combination of rectangles and triangles as in Problem 1.1.4. We can't calculate its area using elementary techniques.

So : what we need is a theory or a method that will allow us to calculate the area bounded by a section of the graph of a function and the x-axis, over a specified interval.

IMPORTANT NOTE: The problem of calculating the distance travelled by an object from knowledge of how its speed is changing is just one example of a scientific problem that can be solved by calculating the area of a region enclosed between a graph and the x-axis. Here are just a few more examples.

1. The fuel consumption of an aircraft is a function of its speed. The total amount of fuel consumed on a journey can be calculated as the area under the graph showing speed against time.
2. The energy stored by a solar panel is a function of the light intensity, which is itself a function of time. The total energy stored in one day can be modelled as the area under a graph of the light intensity against time for that day.
3. The volume of (for example) a square pyramid can be interpreted as the area under a graph of its horizontal cross-section area against height above the base.
4. In medicine, if a drug is administered intravenously, the quantity of the drug that is in the person's bloodstream can be calculated as the area under the graph of a function that depends both on the rate at which the drug is administered and on the rate at which it is processed by the body.
5. The quantity of a pollutant in a lake can be estimated by calculating areas under graphs of functions describing the rate at which the pollutant is being introduced and the the rate it which it is dispersing or being eliminated.
6. The concept of area under a graph is widely used in probability and statistics, where for example the probability that a randomly chosen person is between 1.8m and 1.9m in height is the area under the graph of the appropriate *probability density function*, over the relevant interval.

LEARNING OUTCOMES FOR THIS SECTION

After studying this section, you should be able to

- Explain, with reference to examples, why it is of interest to be able to calculate the area under the graph of a function over some specified interval.
- Explain how such an area can be approximated using rectangles, and how closer approximations can be obtained by taking narrower rectangles and using more of them.
- Solve simple problems similar to Problems 1.1.3 and 1.1.4 in this section.