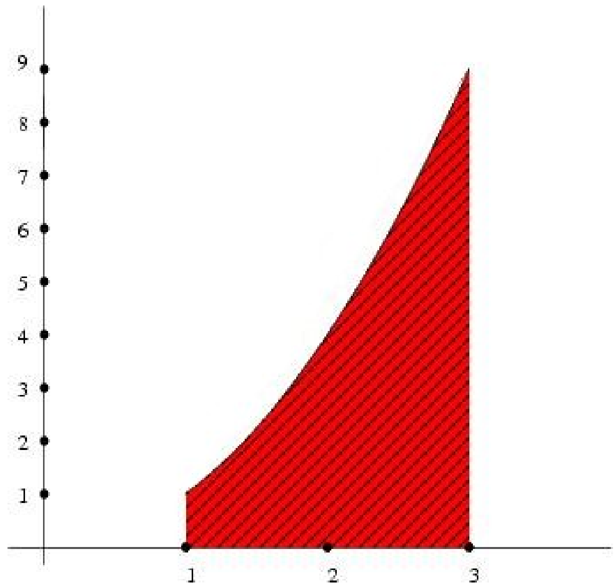
1.2 The Definite Integral

In the last section we concluded that a theory for discussing (and hopefully calculating) areas enclosed between the graphs of known functions and the x-axis, within specified intervals, would be useful. Such a theory does exist and it forms a large part of what is called integral calculus. In order to develop and use this theory we need a technical language and notation for talking about areas under curves. The goal of this section is to understand this notation and be able to use it - it is a bit cumbersome and not the most intuitively appealing, but with a bit of practice it is quite manageable.

Example 1.2.1. *Suppose that f is the function defined by $f(x) = x^2$. Note that $f(x)$ is positive when $1 \leqslant x \leqslant 3$. This means that in the region between the vertical lines $x = 1$ and $x = 5$, the graph $y = f(x)$ lies completely above the $x - \text{axis}$. The area that is enclosed between the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 3$ is called the* definite integral of x^2 from $x = 1$ to $x = 3$, *and denoted by*

$$\boxed{\int_1^3 x^2\,dx.}$$

This diagram shows the region whose area is the definite integral $\int_1^3 x^2\,dx$.



NOTE: At the moment we are not trying to actually *calculate* this red area, we are just thinking about how the integral notation is used and what it means.

Example 1.2.2. *Suppose the function f is defined by*

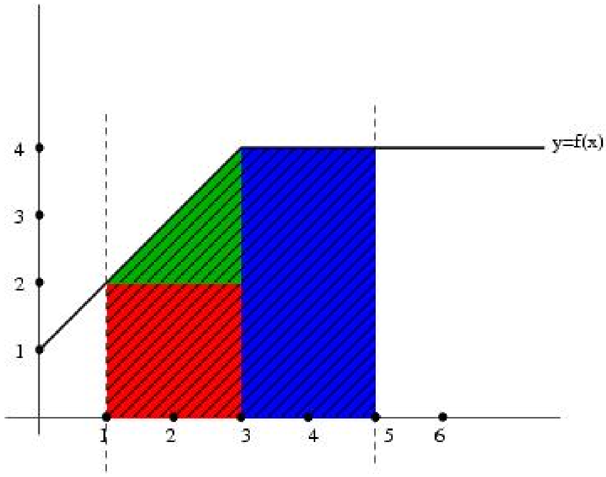
$$f(x) = \begin{cases} x+1 & \text{if } 0 \leqslant x \leqslant 3 \\ 4 & \text{if } x \geqslant 3 \end{cases}$$

Then the graph of f consists of the section of the line $y = x + 1$ between $x = 0$ and $x = 3$ (this is the line segment joining the points $(0, 1)$ and $(3, 4)$, and the constant line $y = 4$ from $x = 3$ onwards.

Now $\displaystyle\int_1^5 f(x)\,dx$ represents the area enclosed by the graph $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. From the diagram below we can see that this area consists of

- A (green) triangle of base length 2 and height 2, area 2;

- A (red) rectangle of base length 2 and height 2, area 4;
- A (blue) rectangle of base length 2 and height 4, area 8.



Adding these three areas, we can conclude that

$$\int_1^5 f(x)\,dx = 2 + 4 + 8 = 14.$$

In this example we are able to calculate the actual value of the definite integral because the region whose area is involved is just an arrangement of rectangles and triangles. Note from this example that in general for a function f and numbers a and b, the definite integral $\int_a^b f(x)\,dx$ is *a number*.

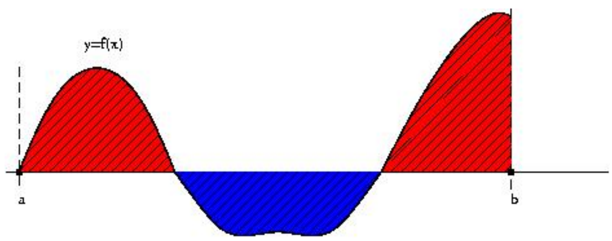
We now move on to the general definition of a definite integral.

Definition 1.2.3. *Let a and b be fixed real numbers, with $a < b$ (so a is to the left of b on the number line). Let f be a function for which it makes sense to talk about the* area *enclosed between the graph of f and the x-axis, over the interval from a to b. Then the* definite integral from a to b of f, *denoted*

$$\int_a^b f(x)\,dx$$

is defined to be the number obtained by subtracting the area enclosed below the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$ from the area enclosed above the x-axis by the graph $y = f(x)$ and the vertical lines $x = a$ and $x = b$.

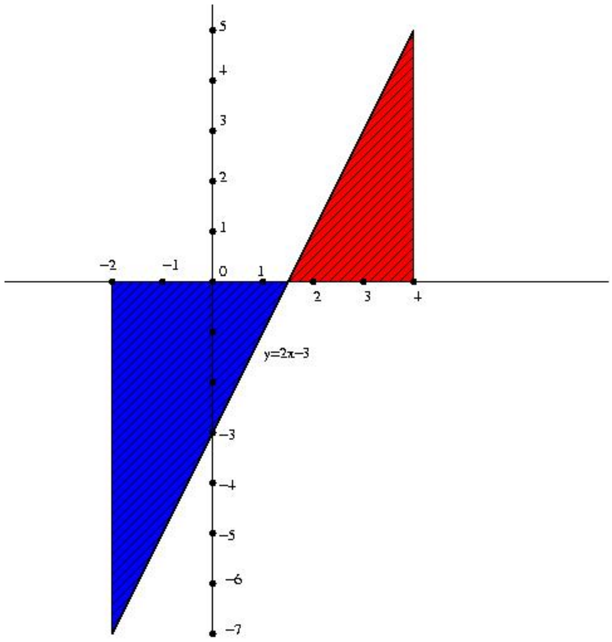
Example 1.2.4. *If the graph $y = f(x)$ is as shown in the diagram below, then $\displaystyle\int_a^b f(x)\,dx$ is the number obtained by subtracting the total area that is coloured blue from the total area that is coloured red.*



Example 1.2.5. *(a) Calculate $\displaystyle\int_{-2}^{4} 2x - 3\,dx$.*
(b) Calculate the total area enclosed between the x-axis and the line $y = 2x-3$, between $x = -2$ and $x = 4$.

SOLUTION: (a) We need to describe the areas enclosed by the curve $y = 2x - 3$, above and below the x-axis, between $x = -2$ and $x = 4$.
The curve $y = 2x-3$ is a line; it passes through the points $(-2, -7)$ and $(4, 5)$ and it intercepts the x-axis at $x = \dfrac{3}{2}$.

The diagram below describes the problem :



The area of the red triangle is

$$\frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4},$$

and the area of the blue triangle is

$$\frac{1}{2} \times \frac{7}{2} \times 7 = \frac{49}{4}.$$

Thus $\displaystyle\int_{-2}^{4} 2x - 3\, dx = \frac{25}{4} - \frac{49}{4} = -\frac{24}{4} = -6.$

(b) The *total area* enclosed between the x-axis and the line $y = 2x - 5$, between $x = -2$ and $x = 4$, is the sum of the areas of the red and blue triangles, which is $\frac{25}{4} + \frac{49}{4} = \frac{37}{2}$.

Note the difference between the two parts of this question, and be careful about this distinction.

NOTES

1. In Definition 1.2.3, What is meant by the phrase "for which it makes sense to talk about the *area* enclosed between the graph of f and the x-axis" is (more or less) that the graph $y = f(x)$ is not just a scattering of points, but consists of a curve or perhaps more than one curve. There is a formal theory about "integrable functions" that makes this notion precise.

2. *Note on Notation*
   The notation surrounding definite integrals is a bit unusual. This note explains the various components involved in the expression

$$\int_a^b f(x)\, dx.$$

- "$\displaystyle\int$" is the *integral sign*.
- The "dx" indicates that f is a function of the variable x, and that we are talking about area between the graph of $f(x)$ against x and the x-axis.
- The "$f(x)$" in $\int_a^b f(x)\,dx$ is called the *integrand*. It is the function whose graph is the upper (or lower) boundary of the region whose area is being described.
- The numbers a and b are respectively called the *lower* and *upper* (or left and right) *limits of integration*. They determine the left and right boundaries of the region whose area is being described.

  In the expression $\int_a^b f(x)\,dx$, the limits of integration a and b are taken to be values of the variable x - this is included in what is to be interpreted from "dx". If there is any danger of ambiguity about this, you can write

  $$\int_{x=a}^{x=b} f(x)\,dx \text{ instead of } \int_a^b f(x)\,dx.$$

  Please do not confuse this use of the word "limit" with its other uses in calculus.

- Important note about this notation : neither the symbol "$\int$" nor the symbol "dx" in this setup is meaningful by itself : they must always accompany each other. You could think of them as being a bit like left and right parentheses or left and right quotation marks - a phrase that is opened with a left parenthesis "(" must be closed by a right parenthesis ")" - neither of these parentheses is meaningful by itself. In the language of definite integrals, an expression that is opened with the integral sign "$\int$" must be closed with the symbol "dx" (or "dt" or "du" as appropriate) indicating the variable involved. The symbol "dx" doesn't really have a meaning by itself - it is a companion to the integral sign.

Some Historical Remarks

The notation that is currently in use for the definite integral was introduced by Gottfried Leibniz around 1675. The rationale for it is as follows :
Areas were estimated as we did in Section 1.1. The interval from a to b would be divided into narrow subintervals, each of width Δx. The name x_i would be given to the left endpoint of the ith subinterval, and the height of the graph above the point x_i would be given by $f(x_i)$. So the area under the graph on this ith subinterval would be approximated by that of a rectangle of width Δx and height $f(x_i)$. The total area would be approximated by the sum of the areas of all of these narrow rectangles, which was written as

$$\sum f(x_i)\Delta x.$$

The accuracy of this estimate improves as the width of the subintervals gets smaller and the number of them gets larger; the true area is the limit of this process as $\Delta x \to 0$. The notation "dx" was introduced as an expression to replace Δx in this limit, and the integral sign $\int$ is a "limit version" of the summation sign $\sum$. The integral symbol itself is based on the "long s" character which was in use in English typography until about 1800.

The idea of calculating areas of regions by taking finer and finer subdivisions in this manner dates back to the ancient Greeks; early examples of what is now called "integration" can be found in the work of Archimedes (circa 225 BC). The idea of computing areas under graphs by taking narrower and narrower vertical columns was put on a firm theoretical basis by Bernhard Riemann in the 1850s.

For more information on the history of calculus and of mathematics generally, see
`http://www-history.mcs.st-and.ac.uk/index.html`.

Learning Outcome for Section 1.2

Just to be able to use the notation for definite integrals correctly and consistently. It is admittedly a bit obscure but give it a bit of careful attention and you will have no problems.