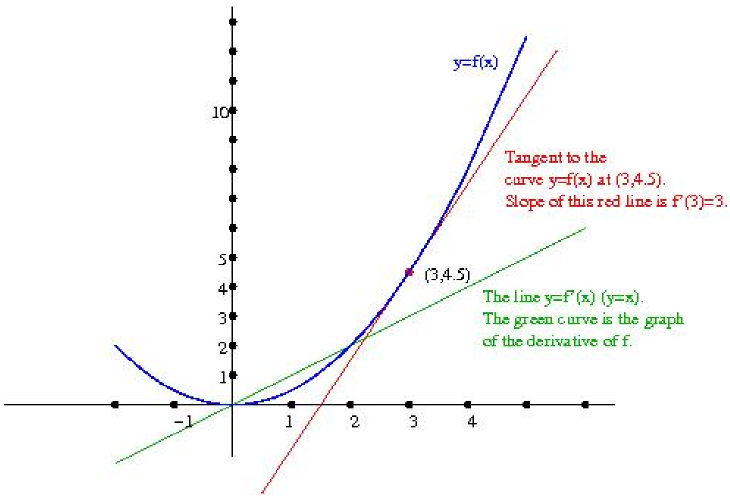
1.3 The Fundamental Theorem of Calculus

In this section, we discuss the *Fundamental Theorem of Calculus* which establishes a crucial link between differential calculus and the problem of calculating definite integrals, or areas under curves. At the end of this section, you should be able to explain this connection and demonstrate with some examples how the techniques of differential calculus can be used to calculate definite integrals.

Differential calculus is about *how functions are changing*. Suppose for example, that you are thinking of temperature (in °C) as a function of time (in hours). You might write temperature as $T(t)$ to indicate that the temperature T varies with time t. The *derivative* of the function $T(t)$, denoted $T'(t)$, tells us how the temperature is *changing over time*. If you know that at 10.00am yesterday the derivative of T was 0.5 (°C/hr), then you know that the temperature was *increasing* by half a degree per hour at that time. However this does not tell you anything about what the temperature actually was at this time. If you know that by 10.00pm last night the derivative of the temperature was -2°C/hr you still don't know anything about what the temperature was at the time, but you know that it was cooling at a rate of 2 degrees per hour. The derivative T' is itself a function of time, as the rate of increase or decrease of temperature will not remain constant throughout the day. Knowing about $T'(t)$ doesn't tell us anything about how warm or cold it was at any given time, but it gives us such information as when it was getting warmer, when it was getting colder, when it stopped getting warmer and started to cool, and so on.

RECALL: Suppose that f is a function of a variable x. Then $f'(x)$ is the *derivative* of f, also a function of x. The value of f' at a particular point a is the *slope* of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$. The diagram below shows the graph of the function defined by $f(x) = \frac{1}{2}x^2$ and the tangent line to this graph at the point (3,4.5). The *slope* of this tangent line (which happens to be 3) is the *derivative* of f when $x = 3$, i.e. it is $f'(3)$. As x varies – as we move along the graph from left to right – the slope of the tangent line varies too, so f' is a function of x; as we know it is given in this example by the formula $f'(x) = \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right) = x$.



Now we are going to define a new function related to definite integrals and consider its derivative - we start with an example.

Example 1.3.1. *At time* $t = 0$ *an object is travelling at 5 metres per second. After* t *seconds its speed in* m/s *is given by*

$$v(t) = 5 + 2t.$$

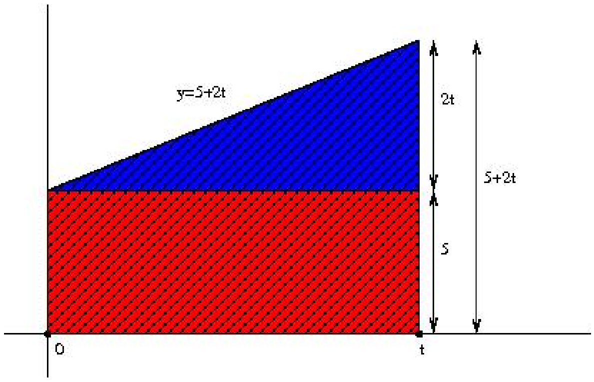
Let $s(t)$ denote the distance travelled by the object after t seconds. So $s(t)$ depends on t obviously since the object is moving over time. From our work in Section 3.1 we know that

$s(t)$ is the area under the graph of $v(t)$ against t, between the vertical lines through 0 and t. We can calculate this in terms of t, by drawing a picture of the graph.

Look at the shape of the region between the graph and the x-axis, between the vertical lines through 0 and t. It is a trapezoid with

- bottom edge formed by a segment of the x-axis of length t;
- left and right edges formed by segments of the vertical lines through 0 and t, of lengths 5 and $5 + 2t$ respectively;
- Top edge formed by part of the graph $y = 5 + 2t$.

The area of this region is $s(t)$. As shown in the diagram, it is the sum of the areas of a rectangle of width t and height 5 (area $5t$) and a triangle of width t and height $2t$ (area t^2). This means: *for any $t \geqslant 0$, the distance covered by this object in the first t seconds of its movement is given by* $s(t) = 5t+t^2$.



IMPORTANT NOTE: The function $s(t)$ associates to t the area under the graph $y = v(t)$ from time 0 to time t. As t increases (i.e. as time passes), this area increases (it represents the distance travelled which is obviously increasing). Note that the derivative of $s(t)$ is exactly $v(t)$.

$$s(t) = 5t + t^2; \; s'(t) = 5 + 2t = v(t).$$

We shouldn't really be surprised by this given the physical context of the problem: $s(t)$ is the total distance travelled at time t, and $s'(t)$ at time t is $v(t)$, the speed at time t. So this is saying that the *instantaneous rate of change* of the distance travelled at a particular moment is the *speed* at which the object is travelling at that moment - which makes sense.

However, there is another way to interpret this statement, which makes sense for definite integrals generally:

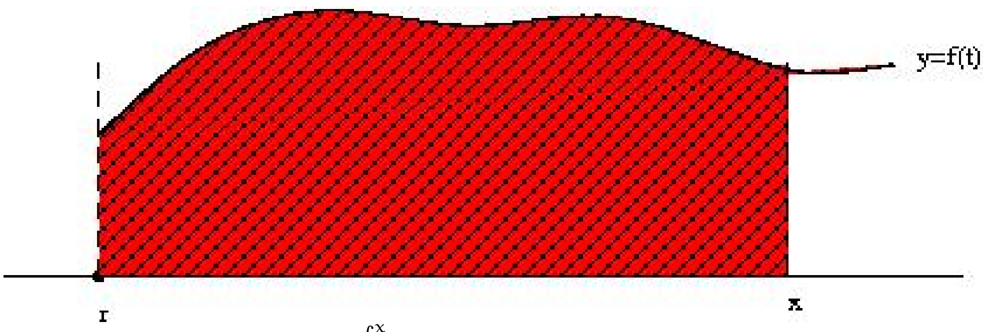
- v is a function whose graph we are looking at.
- For a positive number t, $s(t)$ is the area under the graph of v, to the right of 0 and to the left of t.
- Then the derivative of s is just v, the function under whose graph the area is being measured, i.e $s'(t) = v(t)$.

The more general version of this statement is the *Fundamental Theorem of Calculus*, stated below.

Theorem 1.3.2. *(Fundamental Theorem of Calculus (FToC))*
Let f be a (suitable) function, and let r be a fixed number. Define a function A by

$$A(x) = \int_r^x f(t)\,dt.$$

This means: for a number x, $A(x)$ is the area enclosed by the graph of f and the x-axis, between the vertical lines through r and x. The picture below shows what the function A does.



$$A(x) = \int_r^x f(t)\, dt \text{ is the area shown in red.}$$

The function A *depends on the variable* x*, via the right limit in the definite integral. The Fundamental Theorem of Calculus tells us that the function* f *is exactly the derivative of this area accumulation function* A*. Thus*

$$A'(x) = f(x).$$

Example 1.3.3. *Define a function* F *for* $x \geqslant -6$ *by*

$$F(x) = \int_{-6}^{x} \cos(\pi e^{t^2-4})\, dt.$$

Find $F'(-2)$.

Solution: By the FToC,

$$F'(x) = \cos(\pi e^{x^2-4}), \text{ for } x > -6.$$

Then $F'(-2) = \cos(\pi e^{(-2)^2-4}) = \cos(\pi) = -1$.

NOTES

1. We won't prove the Fundamental Theorem of Calculus, but to get a feeling for what it says, look again at the picture above, and think about how $A(x)$ changes when x moves a little to the right. If $f(x) = 0$, $A(x)$ doesn't change at all as no area is accumulating under the graph of f. If $f(x)$ is positive and large, $A(x)$ increases quickly as x moves to the right. If $f(x)$ is positive but smaller, $A(x)$ increases more slowly with x, because area accumulates more slowly under the "lower" curve. If $f(x)$ is negative, then $A(x)$ will decrease as x increases, because we will be accumulating "negative" area.

2. The Fundamental Theorem of Calculus is *interesting* because it connects differential calculus to the problem of calculating definite integrals, or areas under curves.

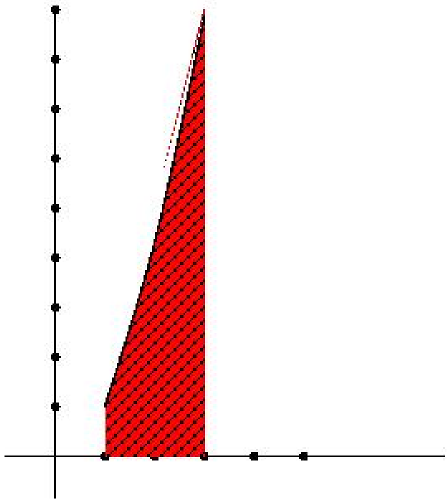
3. The Fundamental Theorem of Calculus is *useful* because we know a lot about differential calculus. Using the machinery of differentiation (the product rule, chain rule etc), we can calculate the derivative of just about anything that can be written in terms of elementary functions (like polynomials, trigonometric functions, exponentials and so on). So we have a lot of theory about differentiation that is all of a sudden relevant to calculating definite integrals as well.

4. The Fundamental Theorem of Calculus can be traced back to work of *Isaac Barrow* and *Isaac Newton* in the mid 17th Century.

Finally we show how to use the Fundamental Theorem of Calculus to calculate definite integrals.

Example 1.3.4. *Calculate* $\int_1^3 t^2\, dt$.

Solution: The area that we want to calculate is shown in the picture below.



Imagine that r is some point to the left of 1, and that the function A is defined for $x \geqslant r$ by

$$A(x) = \int_r^x t^2 \, dt,$$

i.e. $A(x)$ is the area under the graph of t^2 between r and x. Then

$$\int_1^3 x^2 \, dx = A(3) - A(1);$$

this is the area under the graph that is to the left of 3 but to the right of 1. So - if we could calculate $A(x)$, we could evaluate this function at $x = 3$ and at $x = 1$.

What we know about the function $A(x)$, from the Fundamental Theorem of Calculus, is that its derivative is given by $A'(x) = x^2$. What function A has derivative x^2?

The derivative of x^3 is $3x^2$, so the derivative of $\frac{1}{3}x^3$ is x^2.

Note: $\frac{1}{3}x^3$ is *not the only* expression whose derivative is x^2. For example $\frac{1}{3}x^3 + 1$, $\frac{1}{3}x^3 - 5$ and any expression of the form $\frac{1}{3}x^3 + C$ for any constant C, also have derivative x^2. All of these are candidates for $A(x)$: basically they just correspond to different choices for the point r. All of these choices for $A(x)$ give the same outcome when we use them to evaluate $\int_1^3 t^2 \, dt$ as suggested above.

So: take $A(x) = \frac{1}{3}x^3$. Then

$$\int_1^3 x^2 \, dx = A(3) - A(1) = \frac{1}{3}(3^3) - \frac{1}{3}(1^3) = 9 - \frac{1}{3} = \frac{26}{3}.$$

This technique is described in general terms in the following version of the Fundamental Theorem of Calculus:

Theorem 1.3.5. *(Fundamental Theorem of Calculus, Part 2) Let f be a function. To calculate the definite integral*

$$\int_a^b f(x) \, dx,$$

first find a function F whose derivative is f, i.e. for which $F'(x) = f(x)$. (This might be hard). Then

$$\int_a^b f(x) \, dx = F(b) - F(a).$$

Learning outcomes for this section

After studying this section, you should be able to

- Describe what is meant by an "area accumulation function".
- State the Fundamental Theorem of Calculus.
- Use the FToC to solve problems similar to Example 1.3.3.
- Describe the general strategy for calculating a definite integral.
- Evaluate simple examples of definite integrals, like the one in Example 1.3.4.