1.4 Techniques of Integration

Recall the following strategy for evaluating definite integrals, which arose from the Fundamental Theorem of Calculus (see Section 1.3). To calculate

$$\int_a^b f(x)\,dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calculate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $\int_a^b f(x)\,dx$.

Of the three steps above, the first one is the hard one. There are many examples of (very reasonable looking) functions f for which it is not possible to write down a function F whose derivative is f in a manageable way. But there are many also for which it is, and they will be the focus of our attention in this chapter.

Suppose for example we look at the function g defined by $g(x) = \sin(x^2 + x)$. From the chain rule for differentiation we know that $g'(x) = (2x + 1)\cos(x^2 + x)$. But suppose that we started with

$$(2x + 1)\cos(x^2 + x)$$

and we wanted to find something whose derivative with respect to x was equal to this expression. *How would we get back to* $\sin(x^2 + x)$? In this section we will develop answers to this question, but it doesn't have a neat answer. The answer consists of a collection of strategies, techniques and observations that have to be employed judiciously and adapted for each example. It takes some careful practice to become adept at reversing the differentiation process which is basically what we have to do.

Recall the following notation: if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)\big|_a^b \text{ or } F(x)\big|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

We also need the following definition:

Definition 1.4.1. *Let* f *be a function. Another function* F *is called an* antiderivative *of* f *if the derivative of* F *is* f*, i.e. if* $F'(x) = f(x)$*, for all (relevant) values of the variable* x*.*

Thus for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about *an* antiderivative of a function or expression rather that *the* antiderivative. So: a function may have more than one antiderivative, but different antiderivatives of a particular function will always differ from each other by a constant.
Note : Two functions will have the same derivative if their graphs differ from each other only by a vertical shift; in this case the tangent lines to these graphs for particular values of x will always have the same slope.

Definition 1.4.2. *Let* f *be a function. The* indefinite integral *of* f*, written*

$$\int f(x)\,dx$$

is the "general antiderivative" of f*. If* $F(x)$ *is a particular antiderivative of* f*, then we would write*

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the different antiderivatives of f *look like* $F(x) + C$*, where* C *may be any constant. (In this context* C *is often referred to as a constant of integration).*

Example 1.4.3. *We would write*

$$\int 2x \, dx = x^2 + C$$

to indicate that every antiderivative of $2x$ *has the form* x^2+C *for some constant* C*, and that every expression of the form* $x^2 + C$ *(for a constant* C*) has derivative equal to* $2x$*.*

In this section we will consider examples where antiderivatives can be determined without recourse to any sophisticated techniques (which doesn't necessarily mean easily).

The following table reminding us of the derivatives of some elementary functions may be helpful.

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x^2}$	$-\frac{2}{x^3}$
x^n	nx^{n-1}

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\sin 2x$	$2\cos 2x$
e^x	e^x
e^{3x}	$3e^{3x}$

Basically our goal is to figure out how to get from the right to the left column in a table like this.

Example 1.4.4. *Find (i)* $\int x^2 \, dx$*, (ii)* $\int_4^6 x^2 \, dx$

SOLUTION: (i) $\frac{d}{dx}(x^3) = 3x^2$ - so x^3 is not an antiderivative of x^2, it is "too big" by a factor of 3. Thus $\frac{1}{3}x^3$ should be an antiderivative of x^2; indeed

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}3x^2 = x^2.$$

We conclude

$$\int x^2 \, dx = \frac{1}{3}x^3 + C.$$

This means that every antiderivative of x^2 has the form $\frac{1}{3}x^3 + C$ for some constant C.

(ii) By FTC (Part 2) we have

$$\int_4^6 x^2 \, dx = \left.\frac{x^3}{3}\right|_4^6 = \frac{6^3}{3} - \frac{4^3}{3} = \frac{153}{3}.$$

Example 1.4.5. *Determine* $\int \cos 2x \, dx$*.*

SOLUTION: The question is : what do we need to differentiate to get $\cos 2x$? Well, what do we need to differentiate to get something involving cos?
(If you can't answer this question fairly quickly, you are advised to brush up on your knowledge of derivatives of trigonometric functions - don't forget that the SUMS centre can help in this situation).
We know that the derivative of $\sin x$ is $\cos x$.
So a reasonable guess would say that the derivative of $\sin 2x$ might be "something like" $\cos 2x$.
By the chain rule, the derivative of $\sin 2x$ is in fact $2\cos 2x$.
So, in our search for an antiderivative of $\cos 2x$, $\sin 2x$ is pretty close but it gives us twice what we want - we are out by a factor of 2.
So we should compensate for this by taking $\frac{1}{2}\sin 2x$; its derivative is

$$\frac{1}{2}(2\cos 2x) = \cos 2x.$$

CONCLUSION: $\int \cos 2x \, dx = \frac{1}{2}\sin 2x + C.$
NOTE: The reason for the commentary on this example is to give you an idea of the sorts of thought processes a person might go through while figuring out an antiderivative of $\cos 2x$. You would not be expected to provide this sort of commentary if you were answering a question like this in an assessment - it would be enough to just write the line labelled "CONCLUSION" above.
The following examples are similar, with less commentary as we continue.

Example 1.4.6. *Determine* $\int e^{\frac{1}{2}x} \, dx$

SOLUTION: We are looking for something whose derivative is $e^{\frac{1}{2}x}$. We know that the derivative of e^x is e^x, so the answer should be something like $e^{\frac{1}{2}x}$. But this is not exactly right because the derivative of $e^{\frac{1}{2}x}$ is

$$\frac{1}{2}e^{\frac{1}{2}x},$$

which is only half of what we want - we are out by a factor of $\frac{1}{2}$ - what we want is twice what we have. We can compensate for this by multiplying what we have by 2 (or dividing it by $\frac{1}{2}$ which is the same). So what we want is $2e^{\frac{1}{2}x}$ - use the chain rule to confirm that the derivative of this expression is $e^{\frac{1}{2}x}$ as required.
CONCLUSION: $\int e^{\frac{1}{2}x} \, dx = 2e^{\frac{1}{2}x} + C$

Example 1.4.7. *Determine* $\int x^5 \, dx$

SOLUTION: The derivative of x^6 is $6x^5$. So the derivative of $\frac{1}{6}x^6$ is x^5. Hence

$$\int x^5 \, dx = \frac{1}{6}x^6 + C.$$

IMPORTANT NOTE: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an *antiderivative* of x^n we have to reverse this process. This means that the index *increases* by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).
Note: included in the general description of $\int x^n\,dx$ above is the statement that

$$\int 1\,dx = x + C.$$

This makes sense when we ask ourselves what we need to differentiate in order to get 1. The answer is x.

To deal with the exceptional case $n = -1$, suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\dfrac{1}{x}$, and

$$\int \frac{1}{x}\,dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

This latter formula applies for all $x \neq 0$.

Example 1.4.8. *Determine* $\displaystyle\int 3x^2 + 2x + 4\,dx$.

SOLUTION: $\displaystyle\int 3x^2 + 2x + 4\,dx = 3(x^3/3) + 2(x^2/2) + 4x + C = x^3 + 2x^2 + 4x + C.$
Remark: Here we are separately applying our ability to integrate expressions of the form x^n to the x^3 term, the x^2 term, and the constant term. We are also making use of the following fact that indefinite integration behaves *linearly*. This means : if $f(x)$ and $g(x)$ are expressions involving x and a and b are real numbers, we have

$$\int af(x) + bg(x)\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$$

Example 1.4.9. *Determine* $\displaystyle\int_0^{\pi} \sin x + \cos x\,dx$.

SOLUTION: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned}\int_0^{\pi} \sin x + \cos x\,dx &= -\cos x + \sin x|_0^{\pi} \\ &= (-\cos\pi + \sin\pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2.\end{aligned}$$

NOTE: In case you don't find it easy to remember things like cosine and sine of π, $\frac{\pi}{2}$ etc., it is easy enough if you think about it in terms of the definitions of the trigonometric functions. To determine $\cos\pi$, start at the point $(1, 0)$ and travel counter–clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos\pi$, and the y-coordinate is $\sin\pi$.

Example 1.4.10. *Determine* $\displaystyle\int x^{1/3}\,dx$.

SOLUTION: $\displaystyle\int x^{1/3}\,dx = \frac{1}{4/3}x^{4/3} + C = \frac{3}{4}x^{4/3} + C.$

1.4.1 Substitution - Reversing the Chain Rule

The Chain Rule of Differentiation tells us that in order to differentiate the expression $\sin x^2$, we should regard this expression as sin("something") whose derivative (with respect to "something" is cos("something"), then multiply this expression by the derivative of the "something" with respect to x. Thus

$$\frac{d}{dx}(\sin x^2) = \cos x^2 \frac{d}{dx}(x^2) = 2x \cos x^2.$$

Equivalently

$$\int 2x \cos x^2 \, dx = \sin x^2 + C.$$

In this section, through a series of examples, we consider how one might go about reversing the differentiation process to get from $2x \cos x^2$ back to $\sin x^2$.

Example 1.4.11. *Determine* $\int 2x\sqrt{x^2+1}\, dx$.

SOLUTION: Notice that the integrand (i.e. the expression to be integrated) involves both the expressions x^2+1 and $2x$. Note also that $2x$ is the derivative of x^2+1.
Introduce the notation u and set $u = x^2+1$. Note $\dfrac{du}{dx} = 2x$.
Then $2x\sqrt{x^2+1} = \frac{du}{dx}\sqrt{u} = u^{\frac{1}{2}}\dfrac{du}{dx}$.

Suppose we were able to find a function F of u for which $\dfrac{d}{du}(F(u)) = u^{\frac{1}{2}}$. Then by the chain rule we would have

$$\frac{d}{dx}(F(u)) = \frac{d}{du}(F(u))\frac{du}{dx} = u^{\frac{1}{2}}(2x) = 2x\sqrt{x^2+1}.$$

So $F(u)$ would be an antiderivative (with respect to x) of $2x\sqrt{x^2+1}$.
Thus we want

$$\frac{d}{du}(F(u)) = u^{\frac{1}{2}}.$$

So take

$$F(u) = \int u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

(At this stage we are just using the note after Example 1.4.7, with $n = \frac{1}{2}$).
Thus

$$\int 2x\sqrt{x^2+1}\, dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

We usually formulate this procedure of "integration by substitution" in the following more concise way.
To find $\int 2x\sqrt{x^2+1}\, dx$. :
Let $u = x^2+1$.
Then $\dfrac{du}{dx} = 2x \implies du = 2x\, dx$. Then

$$\int 2x\sqrt{x^2+1}\, dx = \int \sqrt{x^2+1}(2x\, dx) = \int u^{\frac{1}{2}}\, du = \frac{2}{3}u^{\frac{3}{2}} + C.$$

So

$$\int 2x\sqrt{x^2+1}\, dx = \frac{2}{3}(x^2+1)^{\frac{3}{2}} + C.$$

Example 1.4.12. *Determine* $\int x\sin(2x^2)\,dx$

SOLUTION: Let $u = 2x^2$.
Then $\dfrac{du}{dx} = 4x\,dx;\ x\,dx = \dfrac{1}{4}du$. So

$$\int x\sin(2x^2)\,dx = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = \frac{1}{4}\cos(2x^2) + C.$$

REMARK: It is good practice to check your answer to a problem like this, either mentally or on paper. Check that the derivative of $-\frac{1}{4}\cos(2x^2)$ is indeed equal to $x\sin(2x^2)$.

Example 1.4.13. *Evaluate* $\displaystyle\int_0^1 \frac{5r}{(4+r^2)^2}\,dr.$

SOLUTION: To find an antiderivative, let $u = 4 + r^2$.
Then $\dfrac{du}{dr} = 2r,\ du = 2r\,dr;\ 5r\,dr = \dfrac{5}{2}du.$
So

$$\int \frac{5r}{(4+r^2)^2}\,dr = \frac{5}{2}\int \frac{1}{u^2}\,du = \frac{5}{2}\int u^{-2}\,du.$$

Thus $\int \dfrac{5r}{(4+r^2)^2}\,dr = -\dfrac{5}{2}\times\dfrac{1}{u} + C$, and we need to evaluate $-\frac{5}{2}\times\frac{1}{u}$ at $r = 0$ and at $r = 1$. We have two choices :

1. Write $u = 4 + r^2$ to obtain

$$\begin{aligned}\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2}\frac{1}{4+r^2}\bigg|_{r=0}^{r=1}\\ &= -\frac{5}{2}\frac{1}{4+1^2} - \left(-\frac{5}{2}\times\frac{1}{4+0^2}\right)\\ &= -\frac{5}{2}\times\frac{1}{5} + \frac{5}{2}\times\frac{1}{4}\\ &= \frac{1}{8}.\end{aligned}$$

2. Alternatively, write the antiderivative as $-\frac{5}{2}\frac{1}{u}$ and replace the limits of integration with the corresponding values of u.
When $r = 0$ we have $u = 4 + 0^2 = 4$.
When $r = 1$ we have $u = 4 + 1^2 = 5$.
Thus

$$\begin{aligned}\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= -\frac{5}{2}\times\frac{1}{u}\bigg|_{u=4}^{u=5}\\ &= -\frac{5}{2}\times\frac{1}{5} - \left(-\frac{5}{2}\times\frac{1}{4}\right)\\ &= \frac{1}{8}.\end{aligned}$$

Example 1.4.14 (Summer paper 2013). *Determine*

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx.$$

Solution: Write

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx.$$

Now write $u = \sqrt{x}+1$. Then

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}} \Longrightarrow \frac{1}{\sqrt{x}}\,dx = 2du.$$

Then

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx &= \int_{x=1}^{x=4} \frac{2}{u}\,du \\ &= \int_{u=2}^{u=3} \frac{2}{u}\,du = 2\ln u|_2^3 \\ &= 2(\ln 3 - \ln 2) = 2\ln\frac{3}{2}.\end{aligned}$$

Note on the exam: This question was answered extremely badly. It was not intended to be particularly difficult or tricky. Only about five people in the whole class answered it correctly. Many candidates made very fundamental and serious errors in algebra before attempting the integration, for example rewriting $\frac{1}{x+\sqrt{x}}$ as $\frac{1}{x}+\frac{1}{\sqrt{x}}$. No credit could be awarded in such a case since the error rendered the rest of the question meaningless. So BE CAREFUL.

Example 1.4.15. *Determine* $\int (1-\cos t)^2 \sin t\,dt$

SOLUTION: Write $u = 1-\cos t$.
Then $\frac{du}{dt} = sint$; $du = \sin t\,dt$.
So

$$\int (1-\cos t)^2 \sin t\,dt = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(1-\cos t)^3 + C.$$

QUESTION: How do we know what expression to extract and refer to as u?
Really what we are doing in this process is changing the integration problem in the variable t to a (hopefully easier) integration problem in a new variable u - there is a change of variables taking place.

There is no easy answer to the question of how to decide what to rename as "u", but with practice we can develop a sense of what might work. In this example the integrand involves the expression $1-\cos t$ and also its derivative $\sin t$. This is what makes the substitution $u = 1-\cos t$ effective for this problem. The "$\sin t$" part of the integrand gets "absorbed" into the "du" in the change of variables, and the "$1-\cos t$" part is obviously easily written in terms of u. We could try the alternative $u = \sin t$, but this is not likely to be helpful, since it is not so easy to see how to express $1-\cos t$ in terms of this u, or what would happen with du which would be effectively $\cos t\,dt$.

Example 1.4.16. *To determine* $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

SOLUTION: How are we to choose u? Well, what are the candidates?
The integrand involves the expressions $1+\sqrt{x}$ and $\frac{1}{\sqrt{x}}$. The derivative of $1+\sqrt{x}$ is "something like" $\frac{1}{\sqrt{x}}$, so setting $u = 1+\sqrt{x}$ might be worth a try.

Let $u = 1+\sqrt{x}$.
Then $\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2}\frac{1}{\sqrt{x}}$; $\frac{1}{\sqrt{x}}dx = 2\,du$.
So

$$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx = 2\int u^3\,du = \frac{2}{4}u^4 + C = \frac{1}{2}(1+\sqrt{x})^4 + C.$$

Example 1.4.17. *Determine* $\int \frac{16x}{\sqrt{8x^2+1}}\, dx$

SOLUTION: Let $u = \sqrt{8x^2+1}$.
Then $\frac{du}{dx} = \frac{1}{2}(8x^2+1)^{-\frac{1}{2}}(16x) = \frac{8x}{\sqrt{8x^2+1}}$.
Thus $\frac{16x}{\sqrt{8x^2+1}}\, dx = 2du$, and

$$\int \frac{16x}{\sqrt{8x^2+1}}\, dx = 2\int du = 2u + C = 2\sqrt{x^2+1} + C.$$

NOTE: An alternative here would have been to set $u = 8x^2+1$. That this would also be successful is left for you to check as an exercise.

Example 1.4.18. *To determine* $\int \frac{\sec^2 x}{\tan x}\, dx$

Note : the derivative of $\tan x$ is $\sec^2 x$, suggesting the substitution $u = \tan x$. You are not necessarily expected to know the derivative of $\tan x$ (or of any of the trigonometric functions) of the top of your head, but you should know where to find them in the "Formulae and Tables" booklet.
Let $u = \tan x$.
Then $\frac{du}{dx} = \sec^2 x$; $du = \sec^2 x\, dx$. Thus $\frac{\sec^2 x}{\tan x}\, dx = \frac{1}{u} du$, and

$$\int \frac{\sec^2 x}{\tan x}\, dx = \int \frac{1}{u}\, du = \log|u| + C = \log|\tan x| + C.$$

1.4.2 Integration by parts - reversing the product rule

In this section we discuss the technique of "integration by parts", which is essentially a reversal of the product rule of differentiation.

Example 1.4.19. *Find* $\int x \cos x \, dx$.

There is no obvious substitution that will help here.
How could $x \cos x$ arise as a derivative?
Well, $\cos x$ is the derivative of $\sin x$. So, if you were differentiating $x \sin x$, you would get $x \cos x$ but according to the product rule you would also get another term, namely $\sin x$. Thus

$$\begin{aligned} \frac{d}{dx}(x \sin x) &= x \cos x + \sin x \\ \Longrightarrow \frac{d}{dx}(x \sin x) - \sin x &= x \cos x. \end{aligned}$$

Note that $\sin x = \frac{d}{dx}(-\cos x)$. So

$$\frac{d}{dx}(x \sin x) - \frac{d}{dx}(-\cos x) = x \cos x \Longrightarrow \frac{d}{dx}(x \sin x + \cos x) = x \cos x.$$

CONCLUSION: $\int x \cos x \, dx = x \sin x + \cos x + C.$

What happened in this example was basically that the product rule was reversed. This process can be managed in general as follows. Recall from differential calculus that if u and v are expressions involving x, then

$$(uv)' = u'v + uv'.$$

Suppose we integrate both sides here with respect to x. We obtain

$$\int (uv)' \, dx = \int u'v \, dx + \int uv' \, dx \Longrightarrow uv = \int u'v \, dx + \int uv' \, dx.$$

This can be rearranged to give the *Integration by Parts Formula* :

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Strategy : when trying to integrate a product, assign the name u to one factor and v' to the other. Write down the corresponding u' (the derivative of u) and v (an antiderivative of v').

The integration by parts formula basically allows us to exchange the problem of integrating uv' for the problem of integrating $u'v$ - which might be easier, if we have chosen our u and v' in a sensible way.

Here is the first example again, handled according to this scheme.

Example 1.4.20. *Use the integration by parts technique to determine* $\int x \cos x \, dx$.

SOLUTION: Write

$$\begin{array}{ll} u = x & v' = \cos x \\ u' = 1 & v = \sin x \end{array}$$

Then

$$\begin{aligned} \int x \cos x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \sin x - \int 1 \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

NOTE: We could alternatively have written $u = \cos x$ and $v' = x$. This would be less successful because we would then have $u' = -\sin x$ and $v = \frac{x^2}{2}$, which looks worse than v'. The integration by parts formula would have allowed us to replace

$$\int x \cos x \, dx \quad \text{with} \quad \int \frac{x^2}{2} \sin x \, dx,$$

which is not an improvement.

So it matters which component is called u and which is called v'.

Example 1.4.21. *To determine* $\int \ln x \, dx$.

SOLUTION: Let $u = \ln x$, $v' = 1$. Then $u' = \frac{1}{x}$, $v = x$.

$$\begin{aligned} \int \ln x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x \ln x - \int \frac{1}{x} x \, dx \\ &= x \ln x - x + C. \end{aligned}$$

NOTE: Example 1.4.21 shows that sometimes problems which are not obvious candidates for integration by parts can be attacked using this technique.

Sometimes two applications of the integration by parts formula are needed, as in the following example.

Example 1.4.22. *To evaluate* $\int x^2 e^x \, dx$.

SOLUTION: Let $u = x^2$, $v' = e^x$. Then $u' = 2x$, $v = e^x$.

$$\begin{aligned} \int x^2 e^x \, dx &= \int uv' \, dx = uv - \int u'v \, dx \\ &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

Let $I = \int x e^x \, dx$.

To evaluate I apply the integration by parts formula a second time.

$$\begin{array}{ll} u = x & v' = e^x \\ u' = 1 & v = e^x. \end{array}$$

Then $I = \int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C$. Finally

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

The next example shows another mechanism by which a second application of the integration by parts formula can succeed where the first is not enough.

Example 1.4.23. *Determine* $\int e^x \cos x \, dx$.

Let
$$\begin{array}{ll} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x. \end{array}$$

Then
$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

For $\int e^x \sin x \, dx$: Let
$$\begin{array}{ll} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x. \end{array}$$

Then
$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx,$$

and
$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right)$$
$$\Longrightarrow 2\int e^x \cos x \, dx \quad = \quad e^x \sin x + e^x \cos x + C$$
$$\Longrightarrow \int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C$$

Finally, an example of a definite integral evaluated using the integration by parts technique.

Example 1.4.24. *Evaluate* $\int_0^1 (x+3)e^{2x} \, dx$.

SOLUTION: Write
$$\begin{array}{ll} u = x+3 & v' = e^{2x} \\ u' = 1 & v = \frac{1}{2}e^{2x} \end{array}$$

Then
$$\begin{aligned} \int_0^1 (x+3)e^{2x} \, dx &= \int uv' \, dx = (uv)\big|_0^1 - \int_0^1 u'v \, dx \\ &= \frac{x+3}{2}e^{2x}\bigg|_0^1 - \frac{1}{2}\int_0^1 e^{2x} \, dx \\ &= \frac{x+3}{2}e^{2x}\bigg|_0^1 - \frac{1}{2} \times \frac{1}{2}e^{2x}\bigg|_0^1 \\ &= \frac{4}{2}e^2 - \frac{3}{2}e^0 - \frac{1}{4}e^2 + \frac{1}{4}e^0 \\ &= \frac{7}{4}e^2 - \frac{5}{4}. \end{aligned}$$

1.4.3 Partial Fraction Expansions - Integrating Rational Functions

We know how to integrate polynomial functions; for example

$$\int 2x^2 + 3x - 4\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x + C.$$

We also know that

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

and that

$$\int \frac{1}{x^n}\,dx = -\frac{1}{n-1}\frac{1}{x^{n-1}} + C,$$

for $n > 1$.

This section is about integrating *rational functions*; i.e. quotients in which the numerator and denominator are both polynomials.

REMARK: If we were presented with the task of adding the expressions $\frac{2}{x+3}$ and $\frac{1}{x+4}$, we would take $(x+3)(x+4)$ as a *common denominator* and write

$$\frac{2}{x+3} + \frac{1}{x+4} = \frac{2(x+4)}{(x+3)(x+4)} + \frac{1(x+3)}{(x+3)(x+4)} = \frac{2(x+4)+1(x+3)}{(x+3)(x+4)} = \frac{3x+11}{(x+3)(x+4)}.$$

Question: Suppose we were presented with the expression $\frac{3x+11}{(x+3)(x+4)}$ and asked to rewrite it in the form $\frac{A}{x+3} + \frac{B}{x+4}$, for *numbers* A and B. How would we do it?

Another Question Why would we want to do such a thing?

Answer to the second question: Maybe if we want to integrate the expression: we know how to integrate things like $\frac{1}{x+3}$, but not things like $\frac{3x+11}{(x+3)(x+4)}$.

Answer to the first question: Write

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A}{x+3} + \frac{B}{x+4}.$$

Then

$$\frac{3x+11}{(x+3)(x+4)} = \frac{A(x+4)}{(x+3)(x+4)} + \frac{B(x+3)}{(x+3)(x+4)} = \frac{(A+B)x+4A+3B}{(x+3)(x+4)}.$$

This means $3x + 11 = (A+B)x + 4A + 3B$ for all x, which means

$$A + B = 3, \text{ and } 4A + 3B = 11.$$

Thus $-4A - 4B = -12,\ -B = -1,\ B = 1$ and $A = 2$. So

$$\frac{3x+11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$

Alternative Method: We want

$$3x + 11 = A(x+4) + B(x+3),$$

for *all* real numbers x. If this statement is true for all x, then in particular it is true when $x = -4$. Setting $x = -4$ gives

$$-12 + 11 = A(0) + B(-1) \Longrightarrow B = 1.$$

Setting $x = -3$ gives
$$-9 + 11 = A(1) + B(0) \Longrightarrow A = 2.$$
Thus
$$\frac{3x + 11}{(x+3)(x+4)} = \frac{2}{x+3} + \frac{1}{x+4}.$$
Expansions of rational functions of this sort are called *partial fraction expansions*.

Example 1.4.25. *Determine* $\int \frac{3x+11}{(x+3)(x+4)}\,dx.$

SOLUTION : Write
$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = \int \frac{2}{x+3}\,dx + \int \frac{1}{x+4}\,dx$$
Then
$$\int \frac{3x+11}{(x+3)(x+4)}\,dx = 2\ln|x+3| + \ln|x+4| + C = \ln(x+3)^2 + \ln|x+4| + C.$$

Example 1.4.26. *Determine* $\int \frac{1}{x^2+5x+6}\,dx.$

SOLUTION: Write $\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)}$ in the form
$$\frac{A}{x+2} + \frac{B}{x+3},$$
for constants A and B. This means
$$\frac{1}{(x+2)(x+3)} = \frac{A(x+3) + B(x+2)}{(x+2)(x+3)},$$
i.e. $1 = A(x+3) + B(x+2)$ for all x.
Thus
$$0x + 1 = (A+B)x + (3A + 2B),$$
which means $A + B = 0$ and $3A + 2B = 1$. This pair of equations has the unique solution $A = 1,\ B = -1$. Thus
$$\begin{aligned} \frac{1}{(x+2)(x+3)} &= \frac{1}{x+2} = \frac{1}{x+3} \\ \Longrightarrow \int \frac{1}{(x+2)(x+3)} &= \int \frac{1}{x+2} - \frac{1}{x+3}\,dx \\ &= \ln|x+3| - \ln|x+2| + C. \end{aligned}$$

NOTE: Any expression of the form $\frac{f(x)}{g(x)}$ where

1. $f(x)$ and $g(x)$ are polynomials and $g(x)$ has higher degree than $f(x)$, and
2. $g(x)$ can be factorized as the product of *distinct* linear factors
$$g(x) = (x - a_1)(x - a_2)\dots(x - a_k)$$

has a *partial fraction expansion* of the form

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_k}{x-a_k},$$

where $A_1,\ A_2,\dots,A_k$ are numbers.

Example 1.4.27. *Determine* $\displaystyle\int \frac{x^3+3x+2}{x+1}\,dx$.

In this example the degree of the numerator exceeds the degree of the denominator, so first apply long division to find the quotient and remainder upon dividing x^3+3x+2 by $x+1$.

We find that the quotient is x^2-x+4 and the remainder is -2. Hence

$$\frac{x^3+3x+2}{x+1} = x^2-x+4+\frac{-2}{x+1}.$$

Thus

$$\int \frac{x^3+3x+2}{x+1}\,dx = \int x^2-x+4\,dx - 2\int \frac{1}{x+1}\,dx = \frac{1}{3}x^3-\frac{1}{2}x^2+4x-2\ln|x+1|+C.$$

NOTE: In the above example we had $\dfrac{f(x)}{g(x)}$ with $f(x)$ of greater degree than $g(x)$. In such cases we can always write

$$\frac{f(x)}{g(x)} = q(x)+\frac{r(x)}{g(x)},$$

where the polynomials $q(x)$ and $r(x)$ are the quotient and remainder respectively on dividing $f(x)$ by $g(x)$, and the degree of $r(x)$ is less than that of $g(x)$.

Example 1.4.28. *Determine* $\displaystyle\int \frac{x+1}{(2x+1)^2(x-2)}\,dx$.

In this case the denominator has a repeated linear factor $2x+1$. It is necessary to include both $\dfrac{A}{2x+1}$ and $\dfrac{B}{(2x+1)^2}$ in the partial fraction expansion. We have

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A}{2x+1}+\frac{B}{(2x+1)^2}+\frac{C}{x-2}.$$

Then

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{A(2x+1)(x-2)+B(x-2)+C(2x+1)^2}{(2x+1)^2(x-2)}.$$

This means that the polynomials $x+1$ and $A(2x+1)(x-2)+B(x-2)+C(2x+1)^2$ are equal, and therefore have the same value when x is replaced by any real number.

$$\begin{array}{rll}
x=2: & 3=C(5)^2 & C=\frac{3}{25}\\[2ex]
x=-\frac{1}{2}: & \frac{1}{2}=B\left(-\frac{5}{2}\right) & B=-\frac{1}{5}\\[2ex]
x=0: & 1=A(1)(-2)+B(-2)+C(1)^2 & A=-\frac{6}{25}
\end{array}$$

Thus

$$\frac{x+1}{(2x+1)^2(x-2)} = \frac{-6/25}{2x+1}+\frac{-1/5}{(2x+1)^2}+\frac{3/25}{x-2}$$

and

$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{6}{25}\int\frac{1}{2x+1}\,dx-\frac{1}{5}\int\frac{1}{(2x+1)^2}\,dx+\frac{3}{25}\int\frac{1}{x-2}\,dx.$$

Call the three integrals on the right above $I_1,\ I_2,\ I_3$ respectively.

- I_1 : Let $u = 2x+1,\ du = 2dx,\ dx = \frac{1}{2}du.$
  $\int \frac{1}{2x+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|(+C_1) = \frac{1}{2}\ln|2x+1|(+C_1).$
- I_2 : Let $u = 2x+1,\ du = 2dx,\ dx = \frac{1}{2}du.$

  $\int \frac{1}{(2x+1)^2}\,dx = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1}(+C_2) = -\frac{1}{2(2x+1)}(+C_2).$
- I_3 : $\int \frac{1}{x-2}\,dx = \ln|x-2|(+C_3).$

Thus
$$\int \frac{x+1}{(2x+1)^2(x-2)}\,dx = -\frac{3}{25}\ln|2x+1| + \frac{1}{10(2x+1)} + \frac{3}{25}\ln|x-2| + C.$$

Learning outcomes for Section 1.4

At the end of this section you should

- Know the difference between a definite and indefinite integral and be able to explain it accurately and precisely.
- Be able to evaluate a range of definite and indefinite integrals using the following methods:
  - direct methods;
  - suitably chosen substitutions;
  - integration by parts;
  - partial fraction expansions.