1.5 Improper Integrals

Suppose that $f(x)$ is a continuous function that satifies

$$\lim_{x\to\infty} f(x) = 0;$$

for example $f(x) = e^{-x}$ has this property. Then we can consider the total area between the graph $y = f(x)$ and the x-axis, to the right of (for example) $x = 1$. This area is denoted by

$$\int_1^\infty f(x)\,dx$$

and referred to as an *improper integral*. For a given function, it is not clear whether the area involved is finite or infinite (if it is infinite, the improper integral is said to *diverge* or to be *divergent*). One question that arises is how we can determine if the relevant area is finite or infinite, another is how to calculate it if it is finite.

Definition 1.5.1. *If the function* f *is continuous on the interval* $[a, \infty)$, *then the* improper integral $\displaystyle\int_a^\infty f(x)\,dx$ *is defined by*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

provided this limit exists. In this case the improper integral is called convergent *(otherwise it's divergent). Similarly, if* f *is continuous on* $(-\infty, a)$, *then*

$$\int_{-\infty}^a f(x)\,dx := \lim_{b\to-\infty} \int_b^a f(x)\,dx$$

Remarks:

1. So to calculate an improper integral of the form $\int_1^\infty f(x)\,dx$ (for example), we first calculate

   $$\int_1^b f(x)\,dx$$

   for a general b. This will typically be an expression involving b. We then take the limit as $b \to \infty$.

2. The condition that $f(x)$ is continuous in the definition above is a bit stronger than we really need. In order to make the definition

   $$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

   what we really need is that $\int_a^b f(x)\,dx$ exists for all b with $b \geqslant a$.

3. If both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ exist for some a, then the improper integral $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ is defined by

   $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

Example 1.5.2. *Show that the improper integral* $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *is divergent.*

Solution:

$$\int_1^b \frac{1}{x}\,dx = \ln x|_1^b = \ln b - \ln 1 = \ln b.$$

Since $\ln b \to \infty$ as $b \to \infty$, $\lim_{b\to\infty} \ln b$ does not exist and the integral diverges.
NOTE: This is related to the divergence of the *harmonic series* $\sum \frac{1}{n}$ which we will discuss later in the course.

Example 1.5.3. *Evaluate* $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^2}\,dx.$

Solution:

$$\int_b^{-2} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_b^{-2} = \frac{1}{2} + \frac{1}{b}$$

Then $\lim_{b\to-\infty}\left(\frac{1}{2}+\frac{1}{b}\right) = \frac{1}{2}$, and

$$\int_{-\infty}^{-2} \frac{1}{x^2}\,dx = \frac{1}{2}.$$

Example 1.5.4. *Determine whether* $\displaystyle\int_1^{\infty} e^{-2x}\,dx$ *is convergent or divergent, and evaluate it if it is convergent.*

Solution: $\displaystyle\int_1^b e^{-2x}\,dx = -\frac{1}{2}e^{-2x}\Big|_1^b = -e^{-2b} + e^{-2}$. Then

$$\int_1^{\infty} e^{-2x}\,dx = \lim_{b\to\infty}(-e^{-2b} + e^{-2}) = e^{-2}.$$

So the integral is convergent and the enclosed area is $\frac{1}{e^2}$.

Example 1.5.5. *Evaluate* $\displaystyle\int_2^{\infty} xe^{-x}\,dx$

Solution: Integrating by parts gives

$$\begin{aligned}\int_2^b xe^{-x}\,dx &= -xe^{-x}\big|_2^b + \int_2^b e^{-x}\,dx \\ &= -be^{-b} + 2e^{-2} - e^{-b} + e^2\end{aligned}$$

Taking the limit as $b \to \infty$, we obtain

$$\int_2^{\infty} xe^{-x}\,dx = \frac{2}{e^2} + e^2.$$

Another type of improper integral

If the graph $y = f(x)$ has a vertical asymptote for a value of x in the interval $[c, d]$, these needs to be considered when computing the integral $\int_c^d f(x)\,dx$, since this integal describes the area of a region that is infinite in the vertical direction at the asymptote.

- If the vertical asymptote is at the left endpoint c, then we define

$$\int_c^d f(x)\,dx = \lim_{b\to c^+}\int_b^d f(x)\,dx,$$

- If the vertical asymptote is at the right endpoint d, then we define

$$\int_c^d f(x)\,dx = \lim_{b\to d^-}\int_c^b f(x)\,dx.$$

- If the vertical asymptote is at an interior point m of the interval $[c, d]$, then we define

$$\int_c^d f(x)\,dx = \int_c^m f(x)\,dx + \int_m^c f(x)\,dx,$$

  and the two improper integrals involving m are handled as above.

As in the case of improper integrals of the first type, these improper integrals are said to *converge* if the limits in question can be evaluated and to diverge if these limits do not exist. Divergence means that the area involved is infinite.

Example 1.5.6. *Determine whether the improper integral* $\int_{-2}^{4} \frac{1}{x^2}\,dx$ *is convergent or divergent.*

SOLUTION: What makes this integral improper is the fact that the graph $y = \frac{1}{x^2}$ has a vertical asymptote at $x = 0$. Thus

$$\int_{-2}^{4} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{4} \frac{1}{x^2}\,dx$$

For the first of these two integrals we have

$$\begin{aligned} \int_{-2}^{0} \frac{1}{x^2}\,dx &= \lim_{b\to 0^-} \int_{-2}^{b} \frac{1}{x^2}\,dx \\ &= \lim_{b\to 0^-} \left(-\frac{1}{x}\right)\Bigg|_{-2}^{b} \\ &= \lim_{b\to 0^-} \left(-\frac{1}{b} + \frac{1}{2}\right) \end{aligned}$$

Since $\lim_{b\to 0^-}(-\frac{1}{b})$ does not exist, the improper integral $\int_{-2}^{0} \frac{1}{x^2}\,dx$ *diverges*. This means that the area enclosed between the graph $y = \frac{1}{x^2}$ and the x-axis over the interval $[-2, 0]$ is *infinite*.

Now that we know that the first of the two improper integrals in our problem diverges, we don't need to bother with the second. The improper integral $\int_{-2}^{4} \frac{1}{x^2}\,dx$ is divergent.