1.6 Exam advice and sample questions for Chapter 1

This section is intended to provide some learning resources for the content of Chapter 1, with a view towards the Summer Exam. The first calculus question on the exam will be based on Chapter 1. The purpose of this question (from the point of view of the examiners) is to assess how well students have achieved the learning outcomes for Chapter 1. These include being able to choose and implement appropriate techniques for evaluating definite, indefinite and improper integrals, as well as being able to discuss the meaning of these terms, the Fundamental Theorem of Calculus, and other theoretical aspects of the subject. There are a few sample exam questions given in this section. The first one is accompanied by a solution that would be considered appropriate and fully correct in an exam context, in terms of the level of explanation and detail provided. In the exam it is important to give a clear and readable account of what you are doing as well as to clearly state your answer.

SAMPLE "EXAM-TYPE" QUESTIONS

1. (a) Define the function A by

$$A(x) = \int_1^x t^3\,dt, \text{ for } x \geqslant 1.$$

   i. Give a geometric or visual description of the meaning of $A(x)$ (you may provide a picture if you wish).
   ii. What does the Fundamental Theorem of Calculus say about the derivative of A?
   iii. What is $A'(3)$?

   (b) Evaluate the following integrals.

$$\text{(i)} \int \cos x\, e^{\sin x}\,dx \quad \text{(ii)} \int_0^\pi x\sin x\,dx \quad \text{(iii)} \int \frac{2x}{x^2+2x-8}\,dx \quad \text{(iv)} \int_2^\infty \frac{x+1}{x^3}\,dx.$$

Solution to Question 1

(a) (i) For a real number $x \geqslant 1$, $A(x)$ is the area enclosed between the graph $y = t^3$ and the horizontal axis, between the vertical lines $t = 1$ and $t = x$.

(ii) $A'(x) = x^3$.

(iii) $A'(3) = 3^3 = 27$.

(b) (i) Write $u = \sin x$. Then $\frac{du}{dx} = \cos x \Longrightarrow \cos x\,dx = du$.
Then

$$\int \cos x\, e^{\sin x}\,dx = \int e^u\,du = e^u + C = e^{\sin x} + C.$$

(ii) Write $u = x,\ v' = \sin x$. Then $u' = 1,\ v = -\cos x$, and using the integration by parts formula we find that

$$\begin{aligned}\int_0^\pi x\sin x\,dx &= -x\cos x\,|_0^\pi + \int_0^\pi \cos x\,dx\\ &= -\pi\cos\pi + \sin x\,|_0^\pi\\ &= \pi.\end{aligned}$$

(iii) Write

$$\frac{2x}{x^2+2x-8} = \frac{2x}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4}.$$

Then $2x = A(x+4) + B(x-2)$. Setting $x = 2$ gives $A = \frac{2}{3}$, and setting $x = -4$ gives $B = -\frac{4}{3}$. Then

$$\int \frac{2x}{x^2+2x-8}\,dx = \frac{2}{3}\int \frac{1}{x-2}\,dx = -\frac{4}{3}\int \frac{1}{x+4}\,dx = \frac{2}{3}\ln|x-2| - \frac{4}{3}\ln|x+4| + C.$$

(iv)

$$\begin{aligned}
\int_2^b \frac{x+1}{x^3} &= \int_2^b \frac{1}{x^2} + \frac{1}{x^3}\, dx \\
&= -\frac{1}{x} \mid_2^b + \left(-\frac{1}{2}1x^2\right) \mid_2^b \\
&= -\frac{1}{b} + \frac{1}{2} + \left(-\frac{1}{2}1b^2 + \frac{1}{2}14\right) \\
&= -\frac{1}{b} - \frac{1}{2b^2} + \frac{5}{8}
\end{aligned}$$

$$\int_2^\infty \frac{x+1}{x^3} = \lim_{b\to\infty}\left(-\frac{1}{b} - \frac{1}{2b^2} + \frac{5}{8}\right) = \frac{5}{8}.$$

2. (a) Explain how the integration by parts formula can be obtained from the product rule for differentiation.

   (b) Evaluate the following integrals.

$$\text{(i)} \int x\cos(x^2+1)\,dx \quad \text{(ii)} \int_0^\pi x\sin\frac{x}{2}\,dx \quad \text{(iii)} \int \frac{x+4}{(x+1)^2}\,dx \quad \text{(iv)} \int_3^\infty x^-3/2\,dx$$

Answers to Question 2 (b)

(i) $\frac{1}{2}\sin(x^2+1)+C$

(ii) 4

(iii) $\ln|x+1| - \frac{3}{x+1}$

(iv) $\frac{2}{3}\sqrt{3}$

3. (a) State the Fundamental Theorem of Calculus and state what this theorem says about the function A defined by

$$A(x) = \int_1^x e^{t^2}\,dt.$$

   (b) Evaluate the following integrals.

$$\text{(i)} \int e^x e^{e^x}\,dx \quad \text{(ii)} \int_0^\pi e^x \sin x\,dx \quad \text{(iii)} \int \frac{x+1}{x^2(x-1)}\,dx \quad \text{(iv)} \int_{-\infty}^1 e^{2x}\,dx$$

Answers to Question 3 (b)

(i) $e^{e^x}+C$

(ii) $\frac{1}{2}(1+e^\pi)$

(iii) $2\ln|x-1| + \frac{1}{x} - 2\ln|x|$

(iv) $\frac{2}{3}\sqrt{3}$