Chapter 2

The Real Numbers

2.1 The set $\mathbb{R}$ of real numbers

This section involves a consideration of properties of the set $\mathbb{R}$ of real numbers, the set $\mathbb{Q}$ of rational numbers, the set $\mathbb{Z}$ of integers and other related sets of numbers. In particular, we will be interested in what is special about $\mathbb{R}$, what distinguishes the real numbers from the rational numbers and why the set of real numbers is such an interesting and important thing that there is a whole branch of mathematics (real analysis) devoted to its study.

WHAT IS $\mathbb{R}$?

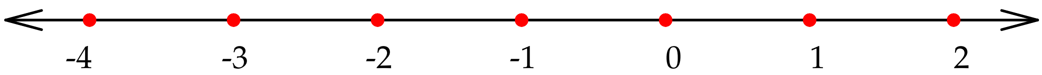
There are at least two useful ways to think about what real numbers are, and before considering them it is useful to first recall what *integers* are and what *rational numbers* are.

Integers are "whole numbers". The set of integers is denoted by $\mathbb{Z}$:

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The notation "$\mathbb{Z}$" comes from the German word *Zahlen* (numbers).

On the number line, the integers appear as an infinite set of evenly spaced points. The integers are exactly those numbers whose decimal representations have all zeroes after the decimal point.



Note that the integers on the number line are separated by *gaps*. For example there are no integers in the chunk of the number line between $\frac{7}{5}$ and $\frac{63}{32}$.

The set of integers is *well-ordered*. This means (more or less) that given any integer, it makes sense to talk about *the next* integer after that one. For example, the next integer after 3 is 4. To see why this property (which might not seem very remarkable at first glance) is something worth bothering about and to understand what it says, it might be helpful to observe that the same property does not hold for the set $\mathbb{Q}$ of *rational numbers* described below.

A *rational number* is a number that can be expressed as a fraction with an integer as the numerator and a non-zero integer as the denominator. The set of all rational numbers is denoted by $\mathbb{Q}$ ($\mathbb{Q}$ is for *quotient*).

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{Z},\ b \neq 0 \right\}.$$

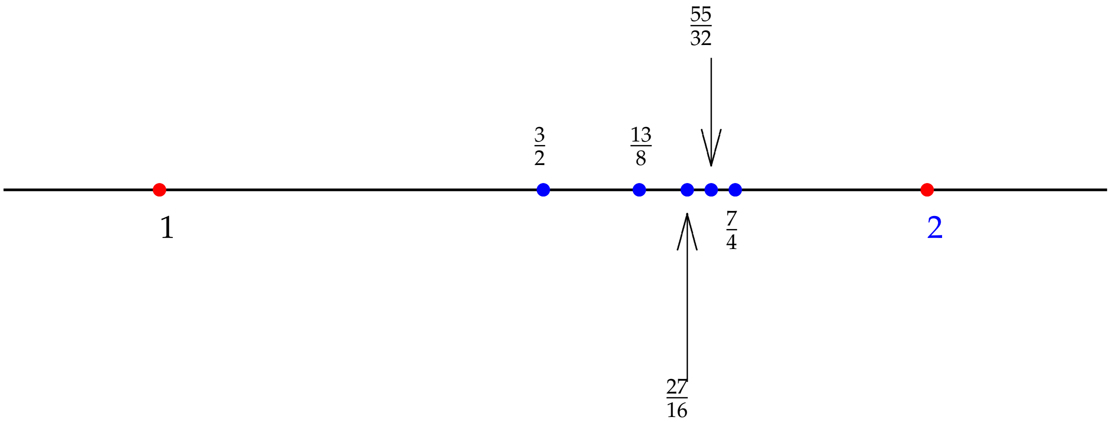
Note: The above statement can be *read* as "*$\mathbb{Q}$ is the set of all numbers that can be written in the form $\frac{a}{b}$ where a is an integer, b is an integer, and b is not zero*". In order to be able to make sense of written mathematics it is essential to be able to read all the parts of statements of this kind and form a

clear mental impression of what is being said. In written mathematics, every mark on the page has meaning and you are expected to attend to that. This takes practice and care and it is not optional if you want to learn mathematics. Mathematical writing is and needs to be entirely unambiguous - this means it has to be technical and fussy, but there is no opportunity for misinterpretation once you are familiar with the relevant definitions and notation.

So $\mathbb{Q}$ includes such numbers as $\frac{5}{7}$, $-\frac{8}{16}$, $\frac{3141}{22445}$ and so on. It includes all the integers, since any integer n can be written in the form of a fraction as $\frac{n}{1}$. The rational numbers are exactly those numbers whose decimal representations either terminate (i.e. all digits are 0 from some point onwards) or repeat (i.e. from some point onwards the decimal part consists of some string of digits repeated over and over without end).

Note: The statement that $\mathbb{Q}$ includes all the integers can be written very concisely as $\mathbb{Z} \subset \mathbb{Q}$. This says: $\mathbb{Z}$ is a subset of $\mathbb{Q}$, i.e. every element of $\mathbb{Z}$ is an element of $\mathbb{Q}$, i.e. every integer is a rational number.

Since rational numbers (by definition) can be written as quotients (or fractions) involving integers, the sets $\mathbb{Q}$ and $\mathbb{Z}$ are closely related. However, on the number line these sets do not resemble each other at all. As mentioned above, the integers are "spaced out" on the number line and there are gaps between them. By contrast, the rational numbers are "densely packed" in the number line. The diagram below is intended to show that the stretch of the number line between 1 and 2 contains infinitely many rational numbers - we can't draw infinitely many of them in a picture, but hopefully this picture indicates how we can keep adding more and more of them indefinitely. By contrast with the situation for $\mathbb{Z}$, there are no stretches of the number line that are without rational numbers.



Note: Does this picture persuade you that there are infinitely many rational numbers between 1 and 2? If not, do you believe this statement? It is up to you to consider its plausibility and reach a conclusion.

Related to these remarks is the observation that the set of rational numbers is not *well-ordered*. Given a particular rational number, there is no *next* rational number after it. For example 0 is a rational number, but there is no *next rational number* after 0 on the number line. This is the same as saying that there is no *smallest* positive rational number. If we had a candidate for this title, we could divide by 2 and we would have a smaller but still positive rational number.

Exercise 2.1.1. *Choose a stretch of the number line - for example the stretch from* $-\frac{7}{4}$ *to* $-\frac{11}{8}$ *(but pick your own example). Persuade yourself that your stretch contains infinitely many rational numbers.*

So the rational numbers are not sparse like the integers. They come close to covering the whole number line in the sense that any stretch (however short) of the number line contains infinitely many rational numbers. The idea of visualizing the points corresponding to rational numbers as a "mist" on the number line has been suggested. However, not all real numbers are rational.

Example 2.1.2. *The number $\sqrt{3}$ is not rational.*

Proof. (by contradiction). Suppose that $\sqrt{3}$ is rational and write

$$\sqrt{3} = \frac{m}{n},$$

where m and n are positive integers with no common integer factors. Then

$$3 = \frac{m^2}{n^2} \Longrightarrow m^2 = 3n^2.$$

This means that m^2 is a multiple of 3, and so m is a multiple of 3, which means that m^2 is actually a multiple of 9; write $m^2 = 9k$, for some $k \in \mathbb{Z}$. Then

$$m^2 = 9k = 3n^2 \Longrightarrow n^2 = 3k,$$

so n^2 is a multiple of 3, hence so is n. But now both m and n are multiples of 3, which means there is no way to write $\sqrt{3}$ in the form $\frac{m}{n}$ for integers m and n with no common factors. □

Now imagine an infinite straight line, on which the integers are marked by an infinite set of evenly spaced dots. Imagine that the rational numbers have also been marked by dots, so that the dot representing $\frac{3}{2}$ is halfway between the dot representing 1 and the dot representing 2, and so on. (Of course it is not physically possible to do all this marking, but it's possible to imagine what the picture would look like). At this stage a lot of dots have been marked - every stretch of the line, no matter how short, contains an infinite number of marked dots.

However, many points on the line remain unmarked. For example, somewhere between the dot representing the rational number 1.4142 and the dot representing the rational number 1.4143 is an unmarked point that represents the real number $\sqrt{2}$. This number is not rational - it cannot be expressed in the form $\frac{a}{b}$ for integers a and b. Somewhere between the dot that marks 3.1415 and the dot that marks 3.1416 is an unmarked point representing the irrational number π. The set $\mathbb{R}$ of *real numbers* is the set of numbers corresponding to *all* points on the line, marked or not. It includes both the rational and irrational numbers.

Note: Because the examples of irrational numbers that are usually cited are things like $\sqrt{2}, \pi$ and e, you could get the impression that irrational numbers are special and rare. This is far from being true. In a very precise way that we will see later, the irrational numbers are more numerous that the rational numbers. If you think of the points representing irrational numbers as a "mist" on the number line, it would be a denser mist than the one for rational numbers. If all the rational numbers were coloured blue on the number line and all the irrational numbers were coloured red, the whole number line would be a jumble of blue and red points, but there would be more red than blue. If you zoomed in, as far as you like, on any section of the number line, however short, both blue and red would still appear, and there would still be more red than blue.

Exercise 2.1.3. *Write down five irrational numbers between 4 and 5.*

Hint: If you don't know what to do, start with $\sqrt{2}$ or some other number that you know is irrational. The number $\sqrt{2}$ is not between 4 and 5 obviously. What adjustments can be made?

Exercise 2.1.4. *Suppose that a is a rational number and b is an irrational number.*

- *Might $a + b$ be irrational?*
- *Must $a + b$ be irrational?*
- *Might ab be irrational?*
- *Must ab be irrational?*
- *Might the product of two irrational numbers be irrational?*

- *Must the product of two irrational numbers be irrational?*

Hint : If in doubt, give yourself some examples and investigate.

To conclude this section we propose two different ways of thinking about the set of real numbers.

1. (*Arithmetic description*) The set $\mathbb{R}$ of real numbers consists of *all* numbers that can be written as (possibly non-terminating and possibly non-repeating) decimals.

   This description is accurate and conceptually it is valuable, but it is not of much practical use because it is not possible to write out a non-terminating non-repeating decimal or do calculations with it. In reality, when we do calculations with decimals (either by hand or by machine), we truncate them at some point and work with approximations which are rational numbers. The set of numbers that a calculator works with is not the set of real numbers or even the set of rational numbers - it is some subset of $\mathbb{Q}$ that depends on the precision of the instrument.

   This arithmetic description of the real numbers highlights the following point. *All* numbers that can be expressed as decimals means *all* numbers that can be written as sequences of the digits $0,1,\ldots,9$ (with a decimal point somewhere) with no pattern of repetition necessary in the digits. In the universe of all such things, the ones that terminate (i.e. end in an infinite string of zeroes) or have a repeating pattern from some point onwards are special and rare. These are the rational numbers. The ones that have all zeroes after the decimal point are even more special - these are the integers.

   Later we will look at the following questions, which might seem at first glance not to even make sense, but which have interesting and surprising answers.

   - Are there more rational numbers than integers?
   - Are there more real numbers than rational numbers?

2. (*Geometric description*) The set $\mathbb{R}$ of real numbers is the set of *all* points on the number line. This is a *continuum* - there are no gaps in the real numbers and no point on the line that doesn't correspond to a real number.

Note: As this course proceeds you will need to know what the symbols $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ mean and be able to recall this information easily. You'll need to be familiar with all the notation involving sets etc. that is used in this section and to be able to use it in an accurate way.

Learning outcomes for Section 2.1

After studying this section you should be able to:

- Use the notation $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ correctly and reliably, and describe the elements of these sets.
- Explain how these sets appear on the number line, and point out some important differences between them.