2.2 Subsets of $\mathbb{R}$

In this section we consider the notions of *finite* and *infinite* sets, and the *cardinality* of a set. Reasonable goals for this section are to become familiar with these ideas and to practice interpreting descriptions of sets that are presented in terse mathematical notation (this means, amongst other things, distinguishing between different kinds of brackets : {}, [ ], ( ), etc.).

Definition 2.2.1. *A set is* finite *if it is possible to list its distinct elements one by one, and this list comes to an end.*
A set is infinite *if any attempt at listing its distinct elements continues indefinitely.*

Example 2.2.2. *The point of this example is not only to show some finite and infinite sets but also to consider the notation that is used to describe them.*

1. $\{1, 2, 3, 4, 5\}$ is a *finite* set - its only elements are the integers 1,2,3,4,5, there are five of them.
2. The interval $[1, 3]$ is an *infinite* set - it consists of *all* the real numbers that are at least equal to 1 and at most equal to 3.
$$[1, 3] := \{x \in \mathbb{R} : 1 \leqslant x \leqslant 3\}.$$
Note: The symbol ":=" here means this is a statement of the *definition* of $[1, 3]$.
3. $\mathbb{Z}$ and $\mathbb{Q}$ are *infinite* sets.
4. The set of real solutions of the equation
$$x^5 + 2x^4 - x^2 + x + 17 = 0$$
is a *finite* set. We don't know how many elements it has, but it has at most five, since each one corresponds to a factor of degree 1 of this polynomial of degree 5.
5. The set of prime numbers is *infinite*.
A pair of *twin primes* is a pair of primes that differ by 2: e.g. 3 and 5, 11 and 13, 59 and 61. It is not known whether the set of pairs of twin primes is finite or infinite.

Definition 2.2.3. *The* cardinality *of a finite set* S*, denoted* $|S|$*, is the number of elements in* S.

Example 2.2.4.

1. If $S = \{5, 7, 8\}$ then $|S| = 3$.
2. $|\{4, 10, \pi\}| = 3$
3. $|\{x \in \mathbb{Z} : \pi < x < 3\pi\}| = 6$.
Note: $\{x \in \mathbb{Z} : \pi < x < 3\pi\} = \{4, 5, 6, 7, 8, 9\}$.
4. The cardinality of $\mathbb{Q}$ is infinite.

Remarks

1. The notation "$|\cdot|$" is severely overused in mathematics. This can be a bit annoying since mathematical text is supposed to be entirely unambiguous. If x is a real number, $|x|$ means the absolute value of x. If S is a set, $|S|$ means the cardinality of S. If A is a matrix, $|A|$ means the determinant of A. There are other usages as well. It is supposed to be clear from the context what is meant.
2. Defining the concept of cardinality for infinite sets is trickier, since you can't say how many elements they have. We will be able to say though what it means for two infinite sets to have the same (or different) cardinalities.

Example 2.2.5. *(A silly example) In a hotel, keys for all the guest rooms are kept on hooks behind the reception desk. If a room is occupied, the key is missing from its hook because the guests have it. If the receptionist wants to know how many rooms are occupied, s/he doesn't have to visit all the rooms to check - s/he can just count the number of hooks whose keys are missing.*

There is nothing deep about this example, but it illustrates a point that is important in mathematics. In the example, the occupied rooms are in *one-to-one correspondence* with the empty hooks. This means that each occupied room corresponds to *one and only one* empty hook, and each empty hook corresponds to *one and only one* occupied room. So the number of empty hooks is the same as the number of occupied rooms and we can count one by counting the other.

Definition 2.2.6. *Suppose that* A *and* B *are sets. Then a* one-to-one correspondence *or a* bijective correspondence *between* A *and* B *is a pairing of each element of* A *with an element of* B*, in such a way that every element of* B *is matched to exactly one element of* A*.*

Definition 2.2.7. *Suppose that* A *and* B *are sets. A function* $f : A \longrightarrow B$ *is called a* bijection *if*

- *Whenever* a_1 *and* a_2 *are* different *elements of* A, $f(a_1)$ *and* $f(a_2)$ *are* different *elements of* B.
- Every *element* b *of* B *is the image of some element* a *of* A.

Note: Definitions 2.2.6 and 2.2.7 are not really much different from each other, but the second one has a bit more technical machinery of a sort that is sometimes useful in trying to describe correspondences. The example about the keys and rooms is a case of both. The sets A and B here are respectively the set of empty hooks and occupied rooms. The bijective correspondence is the matching of each empty hook to the room opened by its key, and the "function" f associates to each hook the corresponding room. The fact that different hooks have different images under f says that each key opens only one room, and the fact that every element of B is the image of *some* element of A says that every occupied room is opened by *some* key belonging to an empty hook.

The translation between the concrete context of Example 2.2.5 and the formal Definition 2.2.7 is completely unnecessary in terms of understanding the example, but the point is that sometimes the objects we are dealing with don't have a concrete context like this and the formal language is actually necessary to describe the situation. The point of the example is just to show that Definition 2.2.7 is not as obscure or as complicated as it might seem at first glance.

Quite often, in order to determine the cardinality of a set, it is easiest to determine the cardinality of another set with which we know it is in bijective correpsondence.

Example 2.2.8. *How many integers between 1 and 1000 are perfect squares?*

Solution: The list of perfect squares in our range begins as follows

$$1,\ 4,\ 9,\ 16, \dots$$

One way of solving the problem would be to keep writing out successive terms of this sequence until we hit one that exceeds 1000, and then delete that one and count the terms that we have. This is actually more work than we are asked to do, since we are not asked for the list of squares but just the number of them.

Alternatively, we could notice that $(31)^2 = 961$ and $(32)^2 = 1024$.
So the numbers $1^2, 2^2, \dots, (31)^2$ are all in the range 1 to 1000 and these are the only perfect squares in that range, the answer to our question is 31.

What we are using here, technically, is the fact that the set of perfect squares in the range of interest is in *bijective correspondence* with the set $\{1, 2, 3, \dots, 31\}$ - the numbers in question are the squares of the first 31 natural numbers. To get the answer 31, it's not really the squares in the range 1 to 1000 that we are counting but the integers in the range 1 to 31.

The following example shows that it could be possible to know that there is a bijective correspondence between two finite sets, without knowing the cardinality of either of them. While this example is a bit contrived, the point of it is to see that sometimes we can show that two sets must be in bijective correspondence even if we don't know much about their elements. This can be a useful device.

Example 2.2.9. *Show that the equations*

$$x^3 + 2x + 4 = 0 \textit{ and } x^3 + 3x^2 + 5x + 7 = 0$$

have the same number of real solutions.

Solution: One way of doing this without having to solve the equations is to demonstrate a bijective correspondence between their sets of real solutions. We can write

$$\begin{aligned} x^3 + 3x^2 + 5x + 7 &= (x^3 + 3x^2 + 3x + 1) + 2x + 6 \\ &= (x+1)^3 + (2x+2) + 4 \\ &= (x+1)^3 + 2(x+1) + 4. \end{aligned}$$

This means that a real number a is a solution of the second equation if and only if

$$(a+1)^3 + 2(a+1) + 4 = 0$$

i.e. if and only if $a + 1$ is a solution of the first equation.

The correspondence $a \longleftrightarrow a+1$ is a bijective correspondence between the solution sets of the two equations. So they have the same number of real solutions.

Note: This number is at least 1 and at most 3. Why?

Learning outcomes for Section 2.2

After studying this section you should be able to

- Explain what is meant by the cardinality of a set;
- Read and interpret descriptions of different subsets of $\mathbb{R}$ presented using different standard notations. Decide what the elements of these sets are and whether the sets are finite or infinite;
- Explain what is meant by a *bijective correspondence* and give examples to support your explanation.