2.3 Infinite sets and cardinality

Recall from the last section that

- The cardinality of a finite set is defined as the number of elements in it.

- Two sets A and B have *the same cardinality* if (and only if) it is possible to match each element of A to an element of B in such a way that every element of each set has exactly one "partner" in the other set. Such a matching is called a *bijective correpondence* or *one-to-one correspondence*. A bijective correspondence between A and B may be expressed as a *function* from A to B that assigns different elements of B to all the elements of A and "uses" all the elements of B. A function that has these properties is called a *bijection*.

In the case of finite sets, the second point above might seem to be overcomplicating the issue, since we can tell if two finite sets have the same cardinality by just counting their elements and noting that they have the same number. The notion of bijective correspondence is emphasized for two reasons. First, as we saw in Example 2.2.9, it is occasionally possible to establish that two finite sets are in bijective correspondence without knowing the cardinality of either of them. More importantly, we would like to develop some notion of cardinality for infinite sets aswell. We can't count the number of elements in an infinite set. However, for a given pair of infinite sets, we could possibly show that it is or isn't possible to construct a bijective correspondence between them.

Definition 2.3.1. *Suppose that A and B are sets (finite or infinite). We say that A and B have the same cardinality (written* $|A| = |B|$*) if a bijective correspondence exists between A and B.*

In other words, A and B have the *same cardinality* if it's possible to match each element of A to a different element of B in such a way that every element of both sets is matched exactly once. In order to say that A and B have *different* cardinalities we need to establish that it's impossible to match up their elements with a bijective correspondence. If A and B are infinite sets, showing that such a thing is *impossible* is a formidable challenge.

The remainder of this section consists of a collection of examples of pairs of sets that have the same cardinality. Recall the following definition.

Definition 2.3.2. *The set* $\mathbb{N}$ *of natural numbers ("counting numbers") consists of all the positive integers.*

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

Example 2.3.3. *Show that* $\mathbb{N}$ *and* $\mathbb{Z}$ *have the same cardinality.*

The sets $\mathbb{N}$ and $\mathbb{Z}$ are both infinite obviously. In order to show that $\mathbb{Z}$ has the same cardinality of $\mathbb{N}$, we need to show that the right-hand column of the table below can be filled in with the integers *in some order*, in such a way that each integer appears there exactly once.

$\mathbb{N}$		$\mathbb{Z}$
1	$\longleftrightarrow$	?
2	$\longleftrightarrow$	?
3	$\longleftrightarrow$	?
4	$\longleftrightarrow$	?
$\vdots$	$\longleftrightarrow$	$\vdots$

So we need to list all the integers on the right hand side, in such a way that every integer appears once. Just following the natural order on the integers won't work, because then there is no first entry for our list. Starting at a particular integer like 0 and then following the natural order won't work, because then we will never get (for example) any negative integers in our list. Something that *will* work is suggested by following the arcs, starting from 0, in the picture below.
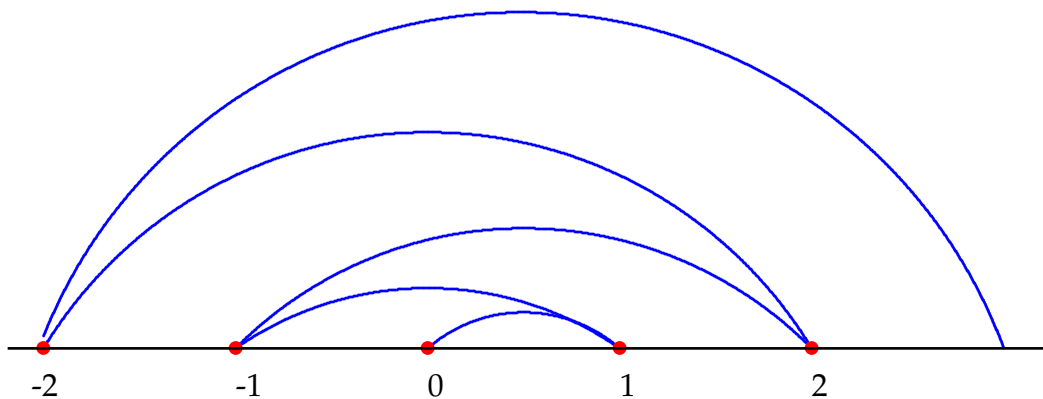
We can start with 0, then list 1 and then -1, then 2 and then -2, then 3 and then -3 and so on. This is a systematic way of writing out the integers, in the sense that given any integer, we can identify the one position where it will appear in our list. For example the integer 10 will be Item 20 in our list, the integer -11 will be Item 22.

Our table becomes

$\mathbb{N}$		$\mathbb{Z}$
1	$\longleftrightarrow$	0
2	$\longleftrightarrow$	1
3	$\longleftrightarrow$	-1
4	$\longleftrightarrow$	2
5	$\longleftrightarrow$	-2
6	$\longleftrightarrow$	3
$\vdots$	$\longleftrightarrow$	$\vdots$

If we want to be fully explicit about how this bijective correspondence works, we can even give a formula for the integer that is matched to each natural number. The correspondence above describes a bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ given for $n \in \mathbb{N}$ by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\[2ex] -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

Exercise 2.3.4. *What integer occurs in position 50 in our list? In what position does the integer -65 appear?*
As well as understanding this example at the informal/intuitive level suggested by the picture above, think about the formula above, and satisfy yourself that it does indeed descibe a bijection between $\mathbb{N}$ and $\mathbb{Z}$. If you are convinced that the two questions above (and all others like them) have unique answers that can be worked out, this basically says that our correspondence between $\mathbb{N}$ and $\mathbb{Z}$ is bijective.

Example 2.3.3 demonstrates a curious thing that can happen when considering cardinalities of infinite sets. The set $\mathbb{N}$ of natural numbers is a *proper subset* of the the set $\mathbb{Z}$ of integers (this means that every natural number is an integer, but the natural numbers do not account for all the integers). Yet we have just shown that $\mathbb{N}$ and $\mathbb{Z}$ are in bijective correspondence. So it is possible for an infinite set to be in bijective correspondence with a proper subset of itself, and hence to have the same cardinality as a proper subset of itself. This can't happen for finite sets (why?).

Putting an infinite set in bijective correspondence with $\mathbb{N}$ amounts to providing a robust and unambiguous scheme or instruction for listing all its elements starting with a first, then a second, third, etc., in such a way that it can be seen that every element of the set will appear exactly once in the list.

Definition 2.3.5. *A set is called* countably infinite *(or* denumerable*) if it can be put in bijective correspondence with the set of natural numbers. A set is called* countable *if it is either finite or countably infinite.*

Basically, an infinite set is countable if its elements can be listed in an inclusive and organised way. "Listable" might be a better word, but it is not really used. Example 2.3.3 shows that the set $\mathbb{Z}$ of integers is countable. To fully appreciate the notion of countability, it is helpful to look at an example of an infinite set that is not countable. This is coming up in Section 2.4.

Thus according to Definition 2.3.1, the sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality. Maybe this is not so surprising, because $\mathbb{N}$ and $\mathbb{Z}$ have a strong geometric resemblance as sets of points on the number line. What is more surprising is that $\mathbb{N}$ (and hence $\mathbb{Z}$) has the same cardinality as the set $\mathbb{Q}$ of all rational numbers. These sets do not resemble each other much in a geometric sense. The natural numbers are sparse and evenly spaced, whereas the rational numbers are densely packed into the number line. Nevertheless, as the following construction shows, $\mathbb{Q}$ is a countable set.

Example 2.3.6. *Show that $\mathbb{Q}$ is countable.*

We need to show that the rational numbers can be organized into a numbered list in a systematic way that includes all of them. Such a list is a one-to-correspondence with the set $\mathbb{N}$ of natural numbers. To construct such a list, start with the following array of fractions.

$$\ldots \quad -3/1 \quad -2/1 \quad -1/1 \quad 0/1 \quad 1/1 \quad 2/1 \quad 3/1 \ \ldots$$

$$\ldots \quad -3/2 \quad -2/2 \quad -1/2 \quad 0/2 \quad 1/2 \quad 2/2 \quad 3/2 \ \ldots$$

$$\ldots \quad -3/3 \quad -2/3 \quad -1/3 \quad 0/3 \quad 1/3 \quad 2/3 \quad 3/3 \ \ldots$$

$$\ldots \quad -3/4 \quad -2/4 \quad -1/4 \quad 0/4 \quad 1/4 \quad 2/4 \quad 3/4 \ \ldots$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

In these fractions, the numerators increase through all the integers as we travel along the rows, and the denominators increase through all the natural numbers as we travel downwards through the columns. Every rational number occurs somewhere in the array. In order to construct a bijective correspondence between $\mathbb{N}$ and the set of fractions in our array, we construct a systematic path that will visit every fraction in the array exactly once. One way of doing this (not the only way) is suggested by the arrows in the following diagram.

$$
\begin{array}{ccccccccccccc}
\ldots & -3/1 & & -2/1 & \leftarrow & -1/1 & & 0/1 & \rightarrow & 1/1 & & 2/1 & \rightarrow & 3/1 & \ldots \\
& \uparrow & & \downarrow & & \uparrow & & & & \downarrow & & \uparrow & & \downarrow & \\
\ldots & -3/2 & & -2/2 & & -1/2 & \leftarrow & 0/2 & \leftarrow & 1/2 & & 2/2 & & 3/2 & \ldots \\
& \uparrow & & \downarrow & & & & & & & & \uparrow & & \downarrow & \\
\ldots & -3/3 & & -2/3 & \rightarrow & -1/3 & \rightarrow & 0/3 & \rightarrow & 1/3 & \rightarrow & 2/3 & & 3/3 & \ldots \\
& \uparrow & & & & & & & & & & & & \downarrow & \\
\ldots & -3/4 & \leftarrow & -2/4 & \leftarrow & -1/4 & \leftarrow & 0/4 & \leftarrow & 1/4 & \leftarrow & 2/4 & \leftarrow & 3/4 & \ldots
\end{array}
$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

This path determines a listing of all the fractions in the array, that starts as follows

$$0/1, \ 1/1, \ 1/2, \ 0/2, \ -1/2, \ -1/1, \ -2/1, \ -2/2, \ -2/3, \ -1/3, \ 0/3, \ 1/3, \ 2/3, \ 2/2, \ 2/1, \ 3/1, \ 3/2, \ 3/3, \ 3/4, \ldots$$

What this example demonstrates is a bijective correspondence between the set $\mathbb{N}$ of natural numbers and the set of all fractions in our array. A bijective correspondence between some infinite set and $\mathbb{N}$ is really just an ordered listing of the elements of that set ("ordered" here just means for the purpose of the list, and the order in which elements are listed doesn't need to relate to any natural order on the set). This is not (exactly) a bijective correspondence between $\mathbb{N}$ and $\mathbb{Q}$.

Exercise 2.3.7. *Why not? (Think about this before reading on.)*

The reason why not is that every rational number appears many times in our array. Already in the section of the list above we have 1/1, 2/2 and 3/3 appearing - these represent different entries in our array but they all represent the same rational number. Equally, the fraction 3/4 appears in the segment of the list above, but 6/8, 9/12 and 90/120 will all appear later.

In order to get a bijective correspondence between $\mathbb{N}$ and $\mathbb{Q}$, construct a list of all the rational numbers from the array as above, but whenever a rational number is encountered that has already appeared, leave it out. Our list will begin

$$0/1, \ 1/1, \ 1/2, \ -1/2, \ -1/1, \ -2/1, \ -2/3, \ -1/3, \ 1/3, \ 2/3, \ 2/1, \ 3/1, \ 3/2, \ 3/4, \ \ldots$$

We conclude that the rational numbers are countable.

Note: Unlike our one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$ in Example 2.3.3, in this case we cannot write down a simple formula to tell us what rational number will be Item 34 on our list (i.e. corresponds to the natural number 34) or where in our list the rational number 292/53 will appear.

In our next example we show that the set of all the real numbers has the same cardinality as an open interval on the real line.

Example 2.3.8. *Show that $\mathbb{R}$ has the same cardinality as the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.*

Note: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \{x \in \mathbb{R} : -\frac{\pi}{2} < x < \frac{\pi}{2}$. The open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ consists of all those real numbers that are strictly between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, not including $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ themselves.

In order to do this we have to establish a bijective correspondence between the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers.

An example of a function that provides us with such a bijective correspondence is familiar from calculus/trigonometry. Recall that for a number x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the number $\tan x$ is defined as follows: travel from $(1,0)$ a distance $|x|$ along the circumference of the unit circle, anti-clockwise if x is positive and clockwise if x is negative. We arrive at a point which is in the right-hand side of the unit circle, because we have travelled a distance of less than $\frac{\pi}{2}$ which would be one-quarter of a full lap. Now $\tan x$ is the ratio of the y-coordinate of this point to the x-coordinate (which are $\sin x$ and $\cos x$ respectively). Equivalently $\tan x$ is the slope of the line joining the origin to our point.

Now $\tan 0 = \dfrac{\sin 0}{\cos 0} = \dfrac{0}{1} = 0$, and as x increases from 0 towards $\frac{\pi}{2}$, $\tan x$ is increasing, since $\sin x$ (the y-coordinate of a point on the circle) is increasing and $\cos x$ (the x-coordinate) is decreasing. Moreover, since $\sin x$ is approaching 1 and $\cos x$ is approaching 0 as x approaches $\frac{\pi}{2}$, $\tan x$ is increasing without limit as x approaches $\frac{\pi}{2}$. Thus the values of $\tan x$ run through *all* the positive real numbers as x increases from 0 to $\frac{\pi}{2}$.

For the same reason, the values of $\tan x$ include *every* negative real number exactly once as x runs between 0 and $-\frac{\pi}{2}$.

Thus for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the correspondence

$$x \longleftrightarrow \tan x$$

establishes a bijection between the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the full set of real numbers. We conclude that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ has the same cardinality as $\mathbb{R}$.
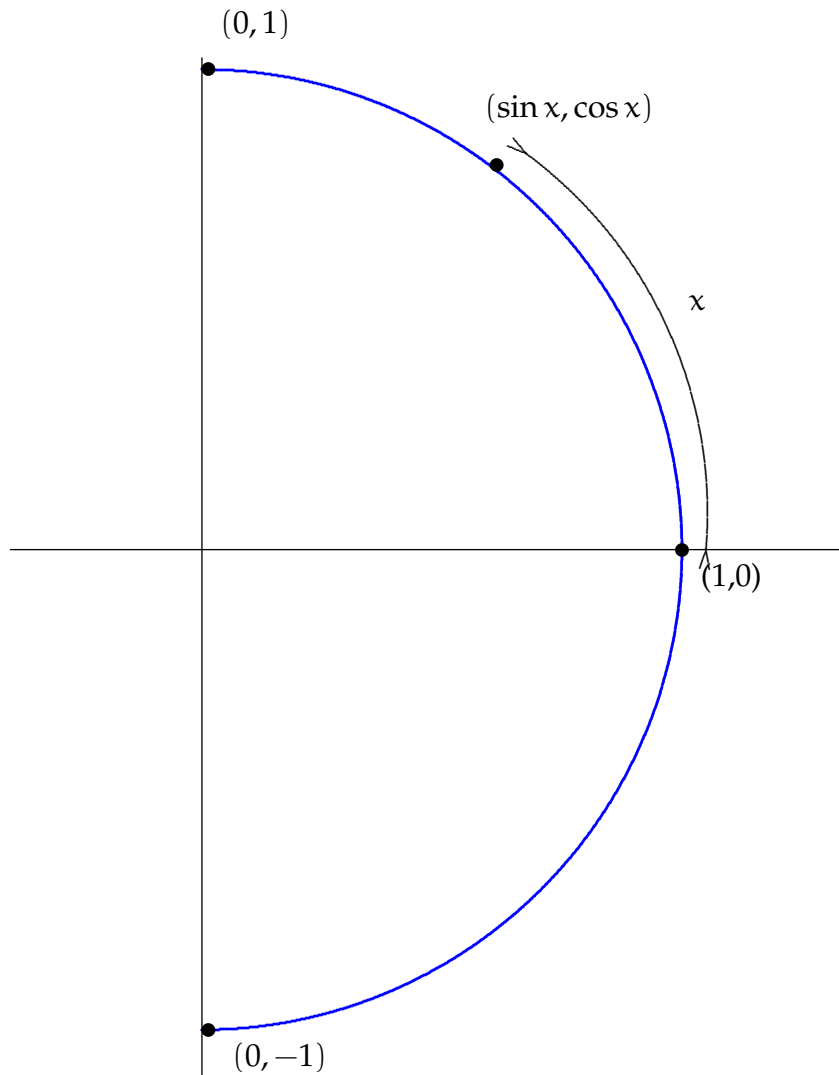
Notes:

1. We don't know yet if $\mathbb{R}$ (or $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) has the same cardinality as $\mathbb{N}$ - we don't know if $\mathbb{R}$ is countable.

2. The interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ may seem like an odd choice for an example like this. A reason for using it in this example is that is convenient for using the tan function to exhibit a bijection with $\mathbb{R}$. However, note that the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is in bijective correspondence with the interval $(-1, 1)$, via the function that just multiplies everything by $\frac{2}{\pi}$.

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \longleftrightarrow \quad (-1, 1)$$

$$x \quad \longleftrightarrow \quad \frac{2x}{\pi}.$$

Exercise 2.3.9. *Show that the the open interval* $(0, 1)$ *has the same cardinality as*

1. *The open interval* $(-1, 1)$

2. *The open interval* $(1, 2)$

3. The open interval $(2, 6)$.

We finish this section now with an account of bounded and unbounded subsets of $\mathbb{R}$. A subset X of $\mathbb{R}$ is *bounded* if, on the number line, its elements do not extend indefinitely to the left or right. In other words there exist real numbers a and b with $a < b$, for which all the points of X are in the interval (a, b).

Definition 2.3.10. *Let X be a subset of $\mathbb{R}$. Then X is* bounded below *if there exists a real number a with $a < x$ for* all *elements x of X. (Note that a need not belong to X here).*
The set X is bounded above *if there exists a real number b with $x < b$ for* all *elements x of X. (Note that b need not belong to X here).*
The set X is bounded *if it is bounded above and bounded below (otherwise it's* unbounded*).*

Example 2.3.11.

1. $\mathbb{Q}$ *is unbounded.*

2. $\mathbb{N}$ *is bounded below but not above.*

3. $(0, 1)$, $[0, 1]$, $[2, 100]$ *are bounded.*

4. $\{\cos x : x \in \mathbb{R}\}$ *is bounded, since* $\cos x$ *can only have values between* -1 *and* 1.

5. *Every finite set is bounded and some infinite sets are.*

Remark: Example 2.3.8 shows that it is possible for a bounded subset of $\mathbb{R}$ to have the same cardinality as the full set $\mathbb{R}$ of real numbers.

LEARNING OUTCOMES FOR SECTION 2.3

This section contains some very challenging concepts. You will probably need to invest some serious intellectual effort in order to arrive at a good understanding of them. This is an effort worth making as it has the potential to really expand your view of what mathematics is about.
After studying this section you should be able to

- Discuss the concept of bijective correspondence for infinite sets;

- Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality by exhibiting a bijective correspondence between them;

- Explain what is meant by a *countable* set and show that $\mathbb{Q}$ is countable;

- Exhibit a bijective correspondence between $\mathbb{R}$ and the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and hence show that $\mathbb{R}$ has the same cardinality as the interval (a, b) for any real numbers a and b with $a < b$.