2.5 The Completeness Axiom in $\mathbb{R}$

The rational numbers and real numbers are closely related, even though the set $\mathbb{Q}$ of rational numbers is countable and the set $\mathbb{R}$ of real numbers is not, and in this sense there are many more real numbers than rational numbers. However, $\mathbb{Q}$ is "dense" in $\mathbb{R}$. This means that every interval of the real number line, no matter how short, contains infinitely many rational numbers. This statement has a practical impact as well, which we use all the time whether consciously or not.

Lemma 2.5.1. *Every real number (whether rational or not) can be approximated by a rational number with a level of accuracy as high as we like.*

Justification for this claim : 3 is a rational approximation for π.
3.1 is a closer one.
3.14 is closer again.
3.14159 is closer still.
3.1415926535 is even closer than that, and we can keep improving on this by truncating the decimal expansion of π at later and later stages. For example if we want a rational approximation that differs from the true value of π by less that 10^{-20} we can truncate the decimal approximation of π at the 21st digit after the decimal point. This is what is meant by "a level of accuracy as high as we like" in the statement of the lemma.

Notes:

1. The fact that all real numbers can be approximated with arbitrary closeness by rational numbers is used all the time in everyday life. Computers basically don't deal with all the real numbers or even with all the rational numbers, but with some specified level of precision. They really work with a subset of the rational numbers.

2. The *sequence*
$$3,\ 3.1,\ 3.14,\ 3.141,\ 3.1415,\ 3.14159,\ 3.141592,\ \dots$$
is a list of numbers that are steadily approaching π. All of these numbers are less than π; they are increasing and they are approaching π. We say that this sequence *converges* to π and we will investigate the concept of convergent sequences in Chapter 3.

3. We haven't looked yet at the question of how the numbers in the above sequence can be calculated, i.e. how we can get our hands on better and better approximations to the value of the irrational number π. That's another thing that we will look at in Chapter 3.

The goal of this last section of Chapter 2 is to pinpoint one essential property of subsets of $\mathbb{R}$ that is not shared by subsets of $\mathbb{Z}$ or of $\mathbb{Q}$. We need a few definitions and some terminology in order to describe this.

Definition 2.5.2. *Let S be a subset of $\mathbb{R}$. An element b of $\mathbb{R}$ is an* upper bound *for S if $x \leqslant a$ for all $x \in S$. An element a of $\mathbb{R}$ is a* lower bound *for S if $a \leqslant x$ for all $x \in S$.*

So an upper bound for S is a number that is to the right of all elements of S on the real line, and a lower bound for S is a number that is to the left of all points of S on the real line. Note that if b is an upper bound for S, then so is every number b' with $b < b'$. If a is a lower bound for S then so is every number a' with $a' < a$. So if S has an upper bound at all it has infinitely many upper bounds, and if S has a lower bound at all it has infinitely many lower bounds. Recall that

- S is *bounded above* if it has an upper bound,
- S is *bounded below* if it has a lower bound,
- S is *bounded* if it is bounded both above and below.

In this section we are mostly interested in sets that are bounded on at least one side.

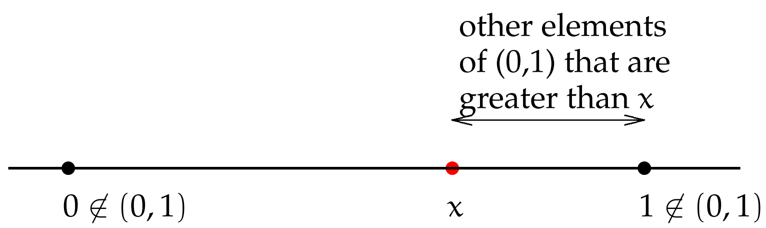
Definition 2.5.3. *Let S be a subset of $\mathbb{R}$. If there is a number m that is both an element of S and an upper bound for S, then m is called the* maximum element *of S and denoted* $\max(S)$.
If there is a number l that is both an element of S and a lower bound for S, then l is called the minimum element *of S and denoted by* $\min(S)$.

Notes

1. A set can have at most one maximum (or minimum) element. For suppose that both m and m' are maximum elements of S according to the definition. Then $m' \leqslant m$ because m is a maximum element of S, and $m \leqslant m'$ because m' is a maximum element of S. The only way that both of these statements can be true is if $m = m'$.

2. Pictorially, on the number line, the maximum element of S is the rightmost point that belongs to S, if such a point exists. The minimum element of S is the leftmost point on the number line that belongs to S, if such a point exists.

3. There are basically two reasons why a subset S of $\mathbb{R}$ might fail to have a maximum element. First, S might not be bounded above - then it certainly won't have a maximum element. Secondly S might be bounded above, but might not contain an element that is an upper bound for itself. Probably the easiest example of this to think about is an open interval like $(0, 1)$. This set is certainly bounded above - for example by 1. However, take any element x of $(0, 1)$. Then x is a real number that is strictly greater than 0 and strictly less than 1. Between s and 1 there are more real numbers all of which belong to $(0, 1)$ and are greater than x. So x is not an upper bound for the interval $(0, 1)$.

   An open interval like $(0, 1)$, although it is bounded, has no maximum element and no minimum element.

An example of a subset of $\mathbb{R}$ that *does* have a maximum and a minimum element is a *closed* interval like $[2, 3]$. The minimum element of $[2, 3]$ is 2 and the maximum element is 3.

Remark : Every *finite* subset of $\mathbb{R}$ has a maximum element and a minimum element (these may be the same if the set has only one element).

For bounded subsets of $\mathbb{R}$, there are notions called the *supremum* and *infimum* that are closely related to maximum and minimum. Every subset of $\mathbb{R}$ that is bounded above has a supremum and every subset of $\mathbb{R}$ that is bounded below has an infimum. This is the *Axiom of Completeness* for $\mathbb{R}$.

Definition 2.5.4. *Let S be a subset of $\mathbb{R}$ that is bounded above. Then the set of all upper bounds for S has a minimum element. This number is called the* supremum *of S and denoted* $\sup(S)$.
Let S be a subset of $\mathbb{R}$ that is bounded below. Then the set of all lower bounds for S has a maximum element. This number is called the infimum *of S and denoted* $\inf(S)$.

Notes

1. The *supremum* of S is also called the *least upper bound (lub)* of S. It is the least of all the numbers that are upper bounds for S.

2. The *infimum* of S is also called the *greatest lower bound (glb)* of S. It is the greatest of all the numbers that are lower bounds for S.

3. Definition 2.5.4 is simultaneously a definition of the terms *supremum* and *infimum* and a statement of the *Axiom of Completeness* for the real numbers.

To see why this statement says something special about the real numbers, temporarily imagine that the only number system available to us is $\mathbb{Q}$, the set of rational numbers. Look at the set

$$S := \{x \in \mathbb{Q} : x^2 < 2\}.$$

So S consists of all those rational numbers whose square is less than 2. It is bounded below, for example by -2, and it is bounded above, for example by 2. This is saying that every rational number whose square is less than 2 is itself between -2 and 2 (of course we could narrow this interval with a bit more care). The positive elements of S are all those positive rational numbers that are less than the real number $\sqrt{2}$.

Claim 1: S does not have a maximum element.

To see this, suppose that x is a candidate for being the maximum element of S. Then x is a rational number and $x^2 \leqslant 2$. For any (very large) integer n, $x + \frac{1}{n}$ is a rational number and

$$\left(x + \frac{1}{n}\right)^2 = x^2 + 2\frac{x}{n} + \frac{1}{n^2}.$$

We can choose n large enough that $2\frac{x}{n} + \frac{1}{n^2}$ is so small that $\left(x + \frac{1}{n}\right)^2$ is still less than 2. Then the number $x + \frac{1}{n}$ belongs to S, and it is bigger than x, contrary to the hypothesis that x could be a maximum element of S.

Claim 2 : S has no least upper bound in $\mathbb{Q}$

To see this, suppose that x is a candidate for being a least upper bound for S in $\mathbb{Q}$. Then $x^2 > 2$. Note x^2 cannot be equal to 2 because $x \in \mathbb{Q}$.

For a (large) integer n

$$\left(x - \frac{1}{n}\right)^2 = x^2 - 2\frac{x}{n} + \frac{1}{n^2} = x^2 - \frac{1}{n}\left(2x - \frac{1}{n}\right).$$

Choose n large enough that $x^2 - \frac{1}{n}\left(2x - \frac{1}{n}\right)$ is still greater than 2. Then $x - \frac{1}{n}$ is still an upper bound for S, and it is less than x.

So S has no least upper bound in $\mathbb{Q}$.

If we consider the same set S as a subset of $\mathbb{R}$, we can see that $\sqrt{2}$ is the supremum of S in $\mathbb{R}$ (and $-\sqrt{2}$ is the infimum of S in $\mathbb{R}$).

This example demonstrates that the *Axiom of Completeness* does not hold for $\mathbb{Q}$, i.e. a bounded subset of $\mathbb{Q}$ need not have a supremum in $\mathbb{Q}$ or an infimum in $\mathbb{Q}$.

Example 2.5.5 (Summer Examinations 2011)**.** *Determine with proof the supremum and infimum of the set*

$$T = \left\{5 - \frac{5}{n} : n \in \mathbb{N}\right\}.$$

Solution: (Supremum) First, look at the numbers in the set. All of them are less than 5. They can be very close to 5 if n is large.

Guess: $\sup(T) = 5$.

We need to show :

1. 5 is an upper bound for T.
   To see this, suppose that $x \in T$. Then $x = 5 - \frac{5}{k}$ for some $k \in \mathbb{N}$. Since k is positive, $\frac{5}{k}$ is positive and $x < 5$. Hence 5 is an upper bound for T.

2. If b is an upper bound for T, then b cannot be less than 5.
   To see this suppose that $b < 5$, so $5 - b$ is a positive real number. We can choose a natural number m so large that $\frac{5}{m} < 5 - b$. Then $5 - \frac{5}{m} > b$, which means that b is not an upper bound for T, as $5 - \frac{5}{m} \in T$.

Thus 5 is the *least upper bound (supremum)* of T.

Solution: (Infimum) Look for the least elements of T. These occur when n is small : when $n = 1$ we get that $5 - \frac{5}{1} = 0$ belongs to T.
Guess: $\inf(T) = 0$.
We need to show :

1. 0 is a lower bound for T.
   To see this, suppose that $x \in T$. Then $x = 5 - \frac{5}{k}$ for some $k \in \mathbb{N}$. Since $k \in \mathbb{N}$, $k \geqslant 1$ and $\frac{5}{k} \leqslant 5$. Thus $5 - \frac{5}{k} \geqslant 5 - 5$ which means $x \geqslant 0$ and 0 is a lower bound for T.

2. If a is a lower bound for T, then a cannot be greater than 0.
   No number greater than 0 can be a lower bound for T, since $0 \in T$. Thus 0 is the *minimum element* (and therefore the infimum) of T.

Learning Outcomes for Section 2.5

After studying this section you should be able to

- State what it means for a subset of $\mathbb{R}$ to be *bounded* (or *bounded above* or *bounded below*).
- Define the terms maximum, minimum, supremum and infimum and explain the connections and differences between them.
- State the Axiom of Completeness.
- Determine whether a set presented like the one in Example 2.5.5 is bounded (above and/or below) or not and identify its maximum/minimum/infimum/supremum as appropriate, with explanation.