2.6 Exam advice and sample questions for Chapter 2

This section is intended to provide some learning resources for the content of Chapter 2, with a view towards the Summer Exam. The second calculus question on the exam will be based on Chapter 2. The purpose of this question (from the point of view of the examiners) is to assess how well students have achieved the learning outcomes for Chapter 2. These learning outcomes are listed at the end of each section of the Lecture notes, and in Chapter 2 they are generally concerned with properties of sets of real numbers such as cardinality, boundedness and countability. There are a few sample exam questions given in this section. The first one is accompanied by a solution that would be considered appropriate and fully correct in an exam context, in terms of the level of explanation and detail provided. In the exam it is important to give a clear and readable account of what you are doing as well as to clearly state your answer.
Don't read too much into these sample questions - they are intended to give you a sense of the style of a typical exam question, but that's all.

SAMPLE "EXAM-TYPE" QUESTIONS

1. *(from the 2013 Summer Exam)*

   (a) For each of the following subsets of $\mathbb{R}$, determine whether it is bounded, countable or neither:
   (i) $\mathbb{R}$ (ii) $\mathbb{Q}$ (iii) $\{x \in \mathbb{Q} : x^2 \leqslant 2\}$ (iv) $\{x \in \mathbb{R} : x^2 < 1\}$.

   (b) By finding a suitable bijection, or otherwise, show that $|(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$. Deduce that $|(a, b)| = |\mathbb{R}|$ for all $a < b \in \mathbb{R}$.

   (c) Determine, with proof, the supremum and infimum of the set
   $$\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\}.$$

Solution

(a) (i) Neither bounded nor countable.
(ii) Countable and not bounded.
(iii) Bounded and countable.
(iv) Bounded and uncountable.

(b) The function tan provides a bijective correspondence between the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$, establishing that these sets have the same cardinality.
Suppose that $a < b$ in $\mathbb{R}$. Then $b - a > 0$ and we may define a bijective correspondence between (a, b) and $(0, b - a)$ by $x \leftrightarrow x - a$. We may define a bijective correspondence bewteen $(0, b - a)$ and $(0, \pi)$ by $x \leftrightarrow \frac{\pi}{b-a}x$. Finally we may define a bijective correspondence between $(0, \pi)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ by $x \leftrightarrow x - \frac{\pi}{2}$. This shows that for all $a < b$ the open interval (a, b) has the same cardinality as $(-\frac{\pi}{2}, \frac{\pi}{2})$ and hence as $\mathbb{R}$.

(c) Write $S = \{\frac{n+1}{n} : n \in \mathbb{N}\}$.
$\sup(S) = 2$. To see this note first that $1 \in S$ since $2 = \frac{1+1}{1}$. Let $x \in S$. Then $x = \frac{m+1}{m} = 1 + \frac{1}{m}$ for some positive integer m. Since $\frac{1}{m} \leqslant 1$, $x \leqslant 1 + 1 \Longrightarrow x \leqslant 2$. Then 2 is both an element of x and an upper bound for S which means $\sup(S) = \max(S) = 2$.
$\inf(S) = 1$. To see this note first that 1 is a lower bound for S, since every element of S has the form $1 + \frac{1}{n}$ for a natural number n and is therefore greater than 1. Suppose that $a > 1$. Then (no matter how close to 1 a is) it is possible to choose a natural number m for which $\frac{1}{m} < a - 1$. Then $1 + \frac{1}{m} < a$, which means that a is not a lower bound for S. Thus 1 is the greatest lower bound of S and $\inf(S) = 1$.

2. (a) Give an example of

    i. a subset of $\mathbb{R}$ that is bounded and uncountable;

    ii. a subset of $\mathbb{R}$ that is countable and bounded above but not below;

    iii. a subset of $\mathbb{R}$ that is bounded and has no maximum element.

   (b) State what is means for an infinite set to be *countable*. Outline Cantor's "diagonal argument" to show that the interval $(0, 1)$ is uncountable.

   (c) Determine, with explanation, the supremum and infimum of the set

$$\left\{ \frac{2}{n^2+1},\ n \in \mathbb{Z} \right\}.$$

3. (a) Determine whether each of the following statements is true or false. Give an explanation for your answer.

    i. The set of irrational real numbers is a countable set.

    ii. Every subset of $\mathbb{Q}$ is countable.

    iii. Every countable subset of $\mathbb{R}$ is a subset of $\mathbb{Q}$.

   (b) Explain what it means for an infinite set to be *countable*. Show that the set of even integers is a countable set.

   (c) Show that the set S defined below is bounded and determine its infimum and supremum, with explanation.

$$S = \left\{ \frac{2n+1}{n^3},\ n \in \mathbb{N} \right\}.$$