Chapter 3

Sequences, Series and Convergence

3.1 Introduction to sequences and series

Example 3.1.1. *Does it make sense to talk about the "number"*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots?$$

What does the question "does it make sense" mean? What we are talking about is the sum of infinitely many specified positive numbers. We can't actually do the addition and calculate what this "number" is based on the definition above. But we can add up any finite collection of the given terms and get an answer for that. Does this sum "settle down" to some identifiable value if we keep adding more terms (whatever that means)? Does it keep growing and growing without bound? Are there ways of finding out? Why would we want to know?

The following experiment might give a slightly vague but hopefully convincing answer to some of these questions. Partially evaluating the sum above for various "initial segments" gives the following results.

- $1 + \frac{1}{4} = 1.25$
- $1 + \frac{1}{4} + \frac{1}{9} \approx 1.361111$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.423611$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10)^2} \approx 1.549767$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(200)^2} \approx 1.639947$
- $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{(10000)^2} \approx 1.644834$

This experiment goes as far as computing the first 10000 terms of the sum, and it appears that the values are not increasing without limit but "settling down" at around 1.6449. What is the significance of this?

$$\frac{\pi^2}{6} \approx 1.644934$$

The *series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges* to the number $\frac{\pi^2}{6}$ (we will have precise definitions for the italicized terms a bit later). This fact is remarkable - there is no obvious connection between π and squares of the form $\frac{1}{n^2}$; moreover all the terms in the series are rational but $\frac{\pi^2}{6}$ is certainly not. This example gives us in principle a way of calculating the digits of π or at least of π^2. (In practice there are similar but better ways, as the convergence in this example is very slow).

Example 3.1.2. *What about*

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots ?$$

Try experimenting with initial segments again :

- The sum of the first four terms is approximately 2.083.
- The sum of the first ten terms is approximately 2.929.
- The sum of the first hundred terms is approximately 5.187.
- The sum of the first thousand terms is approximately 7.485.

There's no sign of this "settling down" or converging to anything that we can identify from this information. This doesn't tell us anything of course - maybe there is convergence but it can't be seen until we take many more millions of terms into our calculation? How could we know that this *doesn't* converge to anything?

Example 3.1.3. *What about*

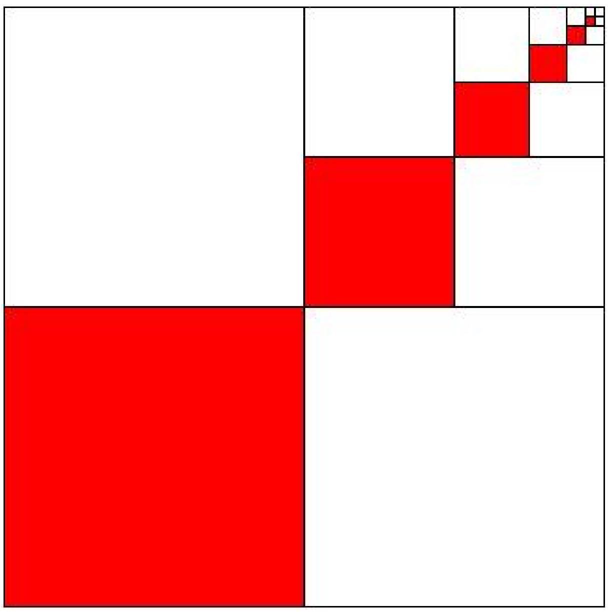
$$\sum_{i=1}^{\infty} \frac{1}{2^{2n}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots ?$$

Experimenting reveals

- $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$
- $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024} = \frac{341}{1024} \approx 0.33301$

These calculations can be verified directly using properties of sums of geometric progressions. It appears that this series is converging to $\frac{1}{3}$.

The following picture gives some graphical evidence for this hypothesis. The large square has area 1, and the red squares have areas $\frac{1}{4}$, $\frac{1}{16}$, etc. The picture is intended to indicate that the red squares occupy one-third of the total area, since every red square is "accompanied" by two white squares of the same area, and all these squares together make up the total area 1. This picture is not really a proof, as it is not possible to actually draw squares representing all the terms of the series, but it is a visual way of understanding the statement that the series $\sum_{n=1}^{\infty} \frac{1}{2^{2n}}$ converges to $\frac{1}{3}$.



We conclude with an example involving the use of a *power series* to represent a function.

Example 3.1.4. *Does it make sense to talk about*

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

as a function of x*?*

If it does, then f must have a domain (consisting of some or all of the real numbers?) and substituting these values in to the definition in place of x must somehow make sense.

- $x = 0 :\ f(0) = 0$
- $x = \frac{\pi}{2} : f(\frac{\pi}{2}) \approx 0.9999$ (six terms)
- $x = \frac{\pi}{6} : f(\frac{\pi}{6}) \approx 0.5000$ (six terms)
- $x = \frac{\pi}{3} : f(\frac{\pi}{3}) \approx 0.8660$ (six terms) ($\frac{\sqrt{3}}{2} \approx 0.8660$)

In all cases we get (just from the first six terms) something very close to $\sin x$. This *series representation* of the sin function will be discussed in a bit more detail later. It provides a means of calculating $\sin x$ for different values of x.

LEARNING OUTCOMES FOR SECTION 3.1

There are no specific learning outcomes listed for Section 3.1 - it is an introduction to the theme of Sequences, Series and Convergence and its purpose is to motivate the theme of convergence and to raise some questions that we will investigate in more detail later in this chapter.