3.2 Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 3.2.1. *A* sequence *is an infinite ordered list*

$$a_1, a_2, a_3, \ldots$$

- The items in list a_1, a_2 etc. are called *terms* (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \ldots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

Example 3.2.2. *1.* $((-1)^n + 1)_{n=1}^{\infty}$ *:* $a_n = (-1)^n + 1$
$a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \ldots$

$$0, 2, 0, 2, 0, 2, \ldots$$

2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$ *:* $a_n = \sin(\frac{n\pi}{2})$
$a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \ldots.$

$$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$ *:* $a_n = \sin(\frac{n\pi}{2})$
$a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \frac{1}{2}\sin(\pi) = 0,\ a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3},\ a_4 = \frac{1}{4}\sin(2\pi) = 0, \ldots.$

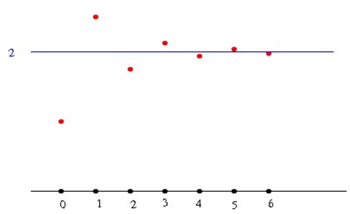
$$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \ldots$$

One way of visualizing a sequence is to consider is as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 3.2.3. $(2 + (-1)^n 2^{1-n})_{n=1}^{\infty}$. *Write* $a_n = 2 + (-1)^n 2^{1-n}$. *Then*

$$a_1 = 2 - 2^0 = 1,\ a_2 = 2 + 2^{-1} = \frac{5}{2},\ a_3 = 2 - 2^{-2} = \frac{7}{4}, a_4 = 2 + 2^{-3} = \frac{17}{8}.$$

Graphical representation of (a_n)*:*



As n gets very large the positive number $\frac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\frac{1}{2^{n-1}}$ as small as we like.

Hence for very large values of n, the number $2 + (-1)^n \frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.

We say that the sequence *converges* to 2, or that 2 is the *limit* of the sequence, and write

$$\lim_{n \to \infty} \left(2 + (-1)^n \frac{1}{2^{n-1}}\right) = 2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

We now state the formal definition of convergence of a sequence. This is reminiscent of the definition of a limit for a function. A sequence converges to the number L if no matter how restrictive your notion of "near L" is, there is a point in the sequence beyond which every term is near L.

Definition 3.2.4. *The sequence* (a_n) converges *to the number* L *(or has* limit L*) if for every positive real number* ε *(no matter how small) there exists a natural number* N *with the property that the term* a_n *of the sequence is within* ε *of* L *for all terms* a_n *beyond the* N*th term. In more compact language :*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon \; \forall n > N.$$

Notes

- If a sequence has a limit we say that it *converges* or *is convergent*. If not we say that it *diverges* or *is divergent*.
- If a sequence converges to L, it means that no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. This means (at least) that beyond a certain point all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret "very close together".

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 3.2.5. *1.* $(\max\{(-1)^n, 0\})_{n=1}^{\infty}$: $0, 1, 0, 1, 0, 1, \ldots$
This sequence alternates between 0 and 1 and does not approach any limit.

2. *A sequence can be divergent by having terms that increase (or decrease) without limit.* $(2^n)_{n=1}^{\infty}$: $2, 4, 8, 16, 32, 64, \ldots$
3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose* n*th term is the* n*th digit after the decimal point in the decimal representation of* π.

Remark: The notion of a convergent sequence is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is *not true* however that the terms in a sequence that converges to a limit L *must* get *progressively* closer to L as n increases, as the following example shows.

Example 3.2.6. *The sequence* a_n *is defined by*

$$a_n = 0 \textit{ if } n \textit{ is even, } a_n = \tfrac{1}{n} \textit{ if } n \textit{ is odd.}$$

This sequence begins :

$$1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \frac{1}{9}, 0, \ldots$$

It converges to 0 *although it is not true that every step takes us closer to zero.*

The following is an example of a convergent sequence.

Example 3.2.7. *Find* $\lim_{n\to\infty} \dfrac{n}{2n-1}$.
Solution*: As if calculating a limit as* $x \to \infty$ *of an expression involving a continuous variable* x*, divide above and below by* n.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

So the sequence $\left(\dfrac{n}{2n-1}\right)$ *converges to* $\frac{1}{2}$.

As for subsets of $\mathbb{R}$, there is a concept of *boundedness* for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below). More precisely :

Definition 3.2.8. *The sequence* (a_n) *is* bounded above *if there exists a real number* M *for which* $a_n \leqslant M$ *for all* $n \in \mathbb{N}$.
The sequence (a_n) *is* bounded below *if there exists a real number* m *for which* $m \leqslant a_n$ *for all* $n \in \mathbb{N}$.
The sequence (a_n) *is* bounded *if it is bounded both above and below.*

Example 3.2.9. *The sequence* (n) *is bounded below (for example by* 0 *or* 1*) but not above. The sequence* $(\sin n)$ *is bounded below (for example by* -1*) and above (for example by* 1*).*

Theorem 3.2.10. *If a sequence is convergent it must be bounded.*

Proof

Note : what we have to do here is use the definitions of convergent and bounded to reason that every sequence that is convergent must be bounded.

Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

(Note: there is nothing special here about $L-1$ *and* $L+1$ *- you could choose* $L-\frac{1}{2}$ *and* $L+\frac{1}{2}$ *or anything like that - the point is that when you choose a certain "window" around* L*, there is a point* (N) *beyond which all the terms of the sequence are in this "window".)*

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.
Let $M = \max\{M_1, L+1\}$ and let $m = \min\{m_1, L-1\}$.

(So M *is defined to be either* M_1 *or* $L+1$*, whichever is the greater, and* m *is defined to be either* m_1 *or* $L-1$*, whichever is the lesser.)*

Then (a_n) is bounded above by M and bounded below by m.
So our sequence is bounded.

Increasing and decreasing sequences

Definition 3.2.11. *A sequence* (a_n) *is called* increasing *if* $a_n \leqslant a_{n+1}$ *for all* $n \geqslant 1$.
A sequence (a_n) *is called* strictly increasing *if* $a_n < a_{n+1}$ *for all* $n \geqslant 1$.
A sequence (a_n) *is called* decreasing *if* $a_n \geqslant a_{n+1}$ *for all* $n \geqslant 1$.
A sequence (a_n) *is called* strictly decreasing *if* $a_n > a_{n+1}$ *for all* $n \geqslant 1$.

Definition 3.2.12. *A sequence is called* monotonic *if it is* either increasing or decreasing.
Similar terms : monotonic increasing, monotonic decreasing, monotonically increasing/decreasing.

Note: These definitions are not *entirely* standard. Some authors use the term *increasing* for what we have called *strictly increasing* and/or use the term *nondecreasing* for what we have called *increasing*.
Examples

1. An increasing sequence is bounded below but need not be bounded above. For example

$$(n)_{n=1}^{\infty} : 1,\ 2,\ 3,\ \ldots$$

2. A bounded sequence need not be monotonic. For example

$$((-1)^n) : -1,\ 1,\ -1,\ 1,\ -1,\ \ldots$$

3. A convergent sequence need not be monotonic. For example

$$\left(\frac{(-1)^{n+1}}{n}\right)_{n=1}^{\infty} : 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$$

This sequence converges to 0 but is neither increasing nor decreasing.

4. A montonic sequence need not be convergent, as Example 1 above shows.

However, if a sequence is *bounded* and *monotonic*, it is *convergent*. This is the *Monotone Convergence Theorem*, which is the major theorem of this section.

Theorem 3.2.13 (The Monotone Convergence Theorem). *If a sequence $(a_n)_{n=1}^{\infty}$ is montonic and bounded, then it is convergent.*

Proof: *(We can start by giving ourselves a monotonic bounded sequence - we can take it to be increasing; the argument for a decreasing sequence is similar.)*
Suppose that (a_n) is *increasing* and bounded. Then the set $(a_n : n \in \mathbb{N})$ is a bounded subset of $\mathbb{R}$ and by the *Axiom of Completeness* it has a *least upper bound* (or supremum) L.

(We are just giving the name L here to the supremum of the set of values of the sequence. We are supposed to be showing that the sequence is convergent, i.e. has a limit : L is our candidate for that limit)

We will show that the sequence (a_n) converges to L.

Choose a (very small) $\varepsilon > 0$. Then $L - \varepsilon$ is *not an upper bound* for $(a_n : n \in \mathbb{N})$, becasue L is the *least* upper bound for this set.

This means there is some $N \in \mathbb{N}$ for which $L - \varepsilon < a_N$. Since L is an upper bound for $(a_n : n \in \mathbb{N})$, this means

$$L - \varepsilon < a_N \leqslant L$$

(i.e. a_N is between $L - \varepsilon$ and L).

Since the sequence (a_n) is increasing and its terms are bounded above by L, *every* term after a_N is between a_N and L, and therefore between $L - \varepsilon$ and L. These terms are all within ε of L.

Using the fact that our sequence is increasing and bounded, we have

- Identified L as the least upper bound for the set of terms in our sequence
- Showed that no matter how small an ε we take, there is a point in our sequence beyond which *all* terms are within ε of L.

This is exactly what it means for the sequence to converge to L. This concludes the proof.

Example 3.2.14 (from 2011 Summer Exam). *A sequence (a_n) is defined by*

$$a_1 = 0, \quad a_{n+1} = \sqrt{a_n + 6} \text{ for all } n \geqslant 1.$$

Show that this sequence is bounded above by 3 and that it is increasing.
Deduce that the sequence is convergent and find its limit.

Note: This is an example of a sequence that is defined *recursively*. This means that the first term is given and subsequent terms are defined (one by one) in terms of previous ones. We are not given a general formula for the nth term although one may exist.

Solution:

1. 3 *is an upper bound.*
   Suppose that $a_k < 3$ for some k. Then

$$a_{k+1} = \sqrt{a_k + 6} < \sqrt{3 + 6} = 3.$$

   This says that if $a_k \leqslant 3$, then $a_{k+1} \leqslant 3$ also.
   Then, since $a_1 < 3$, we have $a_2 < 3$, then $a_3 < 3$, etc.

2. *The sequence is increasing*
   Let $k \in \mathbb{N}$. We need to show that $a_k \leqslant a_{k+1}$.
   We know that $0 \leqslant a_k < 3$: note this implies that

$$a_k = \sqrt{a_k^2} < \sqrt{3a_k} = \sqrt{a_k + 2a_k} < \sqrt{a_k + 6} = a_{k+1}.$$

   Then $a_k < a_{k+1}$ for each k, which means the sequence is increasing.

3. *The sequence converges*
   Since the sequence is increasing and bounded, it converges by the Monotone Convergence Theorem.

   Let L be the limit. Then, taking limits as $n \to \infty$ on both sides of the equation

$$a_{n+1} = \sqrt{a_n + 6}$$

   we find that

$$L = \sqrt{L+6} \Longrightarrow L^2 = L + 6 \Longrightarrow (L-3)(L+2) = 0.$$

   Thus $L = 3$ or $L = -2$, and since all the terms of our sequence are between 0 and 3 it must be that $L = 3$.

Learning Outcomes for Section 3.2

After studying this section you should be able to

- Explain what a sequence is;
- State what it means for a sequence to be
  - convergent or divergent;
  - bounded or unbounded (above or below);
  - monotonic, increasing or decreasing.
- Give and/or identify examples of sequences with or without various properties (or combinations of properties) from the above list;
- State, prove and apply the Monotone Convergence Theorem;
- Analyze examples similar to Example 3.2.14.