3.3 Introduction to Infinite Series

Definition 3.3.1. *A* series *or* infinite *series is the sum of all the terms in a sequence.*

Example 3.3.2 (Examples of infinite series). *1.* $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

2. A geometric *series*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

Every term in this series is obtained from the previous one by multiplying by the common ratio $\frac{1}{2}$. *This is what* geometric *means.*

3. The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

4. An alternating *series*

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

Notes

1. For now these infinite sums are just formal expressions or arrangements of symbols. Whether it is meaningful to think of them as numbers or not is something that can be investigated.

2. A *series* is not the same thing as a *sequence* and it is important not to confuse these terms. A *sequence* is just a list of numbers. A *series* is an infinite sum.

3. The "sigma" notation for sums : *sigma* (lower case σ, upper case Σ) is a letter from the Greek alphabet, the upper case $\sum$ is used to denote sums. The notation

$$\sum_{n=i}^{j} a_n$$

means: i and j are integers and $i \leqslant j$. For each n from i to j the number a_n is defined; the expression above means the sum of the numbers a_n where n runs through all the values from i to j, i.e.

$$\sum_{n=i}^{j} a_n = a_i + a_{i+1} + a_{i+2} + \dots + a_{j-1} + a_j.$$

For example

$$\sum_{n=2}^{5} n^2 = 2^2 + 3^2 + 4^2 + 5^2 = 54.$$

For infinite sums it is possible to have $-\infty$ and/or ∞ (instead of fixed integers i and j) as subscripts and superscripts for the summation.

What does it mean to talk about the sum of *infinitely many* numbers? We cannot add infinitely many numbers together in practice, although we can (in principle) at least, add up any finite collection of numbers. In the examples above we can start from the beginning, adding terms at the start of the series. Adding term by term we get the following lists.

1. $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \dots$

$$1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \dots : 1, 3, 6, 10, 15, \dots$$

   Since the terms being added on at each stage are getting bigger, the numbers in the list above will keep growing (faster and faster as n increases) - we can't associate a numberical value with this infinite sum.

2. A *geometric* series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{2^2}, 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \dots$$

   These numbers are

$$1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32} \dots$$

   In this example the terms that are being added on at each step ($\frac{1}{2^n}$) are getting smaller and smaller as n increases, and the numbers in the list appear to be converging to 2.

3. The *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots \ : \ 1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \dots$$

   It is harder to see what is going on here.

4. An *alternating* series

$$\sum_{n=0}^{\infty} (-1)^n = 1 + (-1) + 1 + (-1) + \dots$$

$$1,\ 1-1,\ 1-1+1,\ 1-1+1-1,\ 1-1+1-1+1 \dots \ : \ 1, 0, 1, 0, 1, \dots$$

   The terms being "added on" at each step are alternating between 1 and -1, and as we proceed with the summation the "running total" alternates between 0 and 1. So there is no numerical value that we can associate with the infinite sum $\sum_{n=0}^{\infty} (-1)^n$.

Note: The series in 2. above *converges* to 2, the series in 1. and 4. are both *divergent* and it is not obvious yet but the series in 3. is *divergent* as well. Our next task is to give precise meanings to these terms for series. In order to do this we need some terminology. Bear in mind that we know what it means for a sequence to converge, but we don't yet have a definition of convergence for series.

Definition 3.3.3. *For a series* $\sum_{n=1}^{\infty} a_n$, *and for* $k \geqslant 1$, *let*

$$s_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \dots + a_k.$$

Thus $s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3$ *etc.*
Then s_k *is called the* kth partial sum *of the series, and the sequence* $(s_k)_{k=1}^{\infty}$ *is called the* sequence of partial sums *of the series.*

If the sequence of partial sums converges to a limit s*, the series is said to* converge *and* s *is called its sum. In this situation we can write*

$$\sum_{n=1}^{\infty} a_n = s.$$

If the sequence of partial sums diverges, *the series is said to* diverge.

Example 3.3.4 (Convergence of a geometric series). *Recall the second example above :*

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

In this example, for $k \geqslant 0$,

$$\begin{aligned} s_k = \sum_{n=0}^{k} \frac{1}{2^n} &= 1 \quad + \quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k} \\ \frac{1}{2} s_k = \sum_{n=1}^{k} \frac{1}{2^{n+1}} &= \qquad\quad \frac{1}{2} + \frac{1}{4} + \dots \frac{1}{2^k} + \frac{1}{2^{k+1}} \end{aligned}$$

Then

$$s_k - \frac{1}{2} s_k = \frac{1}{2} s_k = 1 - \frac{1}{2^{k+1}} \Longrightarrow s_k = 2 - \frac{1}{2^k}.$$

So the *sequence of partial sums* has kth term $2 - \frac{1}{2^k}$. This sequence converges to 2 so the series converges to 2; we can write

$$\sum_n = 0^{\infty} \frac{1}{2^k} = 2.$$

General geometric series : Consider the sequence of partial sums for the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \dots$$

(This is a *geometric series* with initial term a and *common ratio* r.) The kth partial sum s_k is given by

$$\begin{aligned} s_k = \sum_{n=0}^{k} ar^n &= a \quad + \quad ar + \dots ar^k \\ rs_k = \sum_{n=0}^{k} ar^{n+1} &= \qquad\quad ar + ar^2 + \dots ar^{k+1} \end{aligned}$$

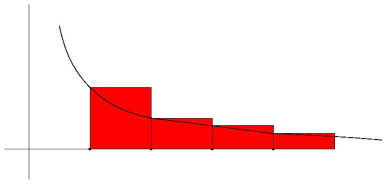
Then $(1-r)s_k = a - ar^{k+1} \Longrightarrow s_k = \dfrac{a(1-r^{k+1})}{1-r}$. If $|r| < 1$, then $r^{k+1} \to 0$ as $k \to \infty$, and the sequence of partial sums (hence the series) converges to $\dfrac{a}{1-r}$. If $|r| \geqslant 1$ the series is divergent.

Next we show that the harmonic series is divergent.

Theorem 3.3.5. *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

We give two proofs.

Proof 1: Think of $\frac{1}{n}$ as the area of a rectangle of height $\frac{1}{n}$ and width 1, sitting on the interval $[n, n+1]$ on the x-axis. So the first term $\frac{1}{1}$ corresponds to a square of area 1 sitting on the interval $[1, 2]$, the term $\frac{1}{2}$ corresponds to a rectangle of area $\frac{1}{2}$ sitting on the interval $[2, 3]$ and so on, as in the following picture.



The total area accounted for by these triangles is the sum of the harmonic series, and this exceeds the area accounted for by the improper integral

$$\int_1^\infty \frac{1}{x}\,dx.$$

From Section 1.5 we know that this area is infinite, hence the series is divergent.

Proof 2: We show that the sequence of partial sums of the harmonic series is not bounded above.

- The first term is 1.
- The second term is $\frac{1}{2}$.
- The sum of the 3rd and 4th terms exceeds $\frac{1}{2}$:

$$\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}.$$

- The sum of the 5th, 6th, 7th and 8th terms exceeds $\frac{1}{2}$:

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}.$$

- For the same reason, the sum of the next 8 terms (terms 9 through 16) also exceeds $\frac{1}{2}$.
- In general the sum of the 2^{n-1} terms $\dfrac{1}{2^{n-1}+1}$ through $\dfrac{1}{2^n}$ exceeds $\frac{1}{2}$.

So, as we list terms in the sequence of partial sums of the harmonic series, we keep accumulating non-overlapping stretches of terms that add up to more than $\frac{1}{2}$. Thus the entire series has infinitely many non-overlapping stretches all individually summing to more than $\frac{1}{2}$. Then the sum of this series is not finite and the series diverges.

Note: A necessary condition for the series $\sum_{n=1}^\infty a_n$ to *converge* is that the *sequence* $\{a_n\}_{n=1}^\infty$ converges to 0; i.e. that $a_n \to 0$ as $n \to \infty$. If this does *not* happen, then the sequence of partial sums has no possibility of converging.

The example of the harmonic series shows that the condition $a_n \to 0$ as $n \to \infty$ is not *sufficient* to guarantee that the series $\sum_{n=1}^\infty$ will converge.

LEARNING OUTCOMES FOR SECTION 3.3

After studying this section you should be able to

- explain what an infinite series is and what it means for an infinite series to converge;
- Give examples of convergent and divergent series;
- show that the harmonic series is divergent;
- Use the "sigma" notation for sums.