

3.4 Introduction to power series

Definition 3.4.1. A polynomial in the variable x is an expression of the form

$$\sum_{i=0}^n a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

or $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$

where the coefficients a_i are real numbers and n is a natural number. The degree of the polynomial is the highest k for which x^k appears with non-zero coefficient.

Examples

1. $x^3 + 5x^2$ is a polynomial of degree 3 (also called cubic).
2. $2 + 2x + 2x^2 + 2x^7$ is a polynomial of degree 7.
3. $x^4 + x^3 + \frac{3}{x^2}$ is *not* a polynomial, because it involves a negative power of x .

The point is that a polynomial can have a constant term (which may be zero) and a *finite* number of terms involving particular positive powers of x that have numbers as coefficients. A polynomial may be regarded as a function of x , and polynomials are functions of a special type.

Definition 3.4.2. A power series in the variable x resembles a polynomial, except that it may contain infinitely many positive powers of x . It is an expression of the type

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots,$$

where each a_i is a number.

Example 3.4.3.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

is a power series.

Question: Does it make sense to think of a power series as a *function* of x ? We investigate this question for the example above.

So define a “function” by

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

- If we try to evaluate this function at $x = 2$, we get a *series* of real numbers.

$$f(2) = \sum_{n=0}^{\infty} 2^n = 1 + 2 + 2^2 + \cdots$$

This series is divergent, so our power series does not define a function that can be evaluated at 2.

- If we try evaluating at 0 (and allow that the first term x^0 of the power series is interpreted as 1 for *all* values of x), we get

$$f(0) = 1 + 0 + 0^2 + \cdots = 1.$$

So it does make sense to “evaluate” this function at $x = 0$.

- If we try evaluating at $x = \frac{1}{2}$, we get

$$f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots$$

From our work on geometric series in Section 2.3 we know that this is a geometric series with first term $a = 1$ and common ratio $r = \frac{1}{2}$. We know that if $|r| < 1$, such a series converges to the number $\frac{a}{1-r}$. In this case

$$\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2,$$

and we have $f\left(\frac{1}{2}\right) = 2$.

In general we know that a geometric series of this sort converges provided that the absolute value of its common ratio is less than 1. So for example if we put $x = \frac{1}{3}$ we find that $f\left(\frac{1}{3}\right)$ is the sum of a geometric series with first term 1 and common ratio $\frac{1}{3}$; this is

$$\frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

In general for any value of x whose absolute value is less than 1 (i.e. any x in the interval $(-1, 1)$), we find that $f(x)$ is a convergent geometric series, converging to $\frac{1}{1-x}$.

Conclusion: For values of x in the interval $(-1, 1)$ (i.e. $|x| < 1$), the function $f(x) = \frac{1}{1-x}$ coincides with the power series $\sum_{n=0}^{\infty} x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \text{ for } |x| < 1.$$

The interval $(-1, 1)$ is called the *interval of convergence* of the power series, and 1 is the *radius of convergence*. We say that the *power series representation* of the function $f(x) = \frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$, for values of x in the interval $(-1, 1)$.

Note: The expression $\frac{1}{1-x}$ makes sense for all values of x except $x = 1$. We are not saying that the domain of the function $f(x) = \frac{1}{1-x}$ only consists of the interval $(-1, 1)$, but just that it is only on this interval that our power series represents this function.

Remark: The fact that for certain values of x we can represent $\frac{1}{1-x}$ with a power series might be interesting (at least to some people!), but it is not of particular use if you want to calculate $\frac{1}{1-x}$ for some particular value of x , because this is easily done directly. However, if we could obtain a power series representation for a function like $\sin x$ and use it to evaluate (or approximate) $\sin(1)$ or $\sin(9)$ or $\sin(20)$, that might be of real practical use. These numbers are *not* easy to obtain directly because the definition of $\sin x$ doesn't tell us how to calculate $\sin x$ for a particular x - you can use a calculator of course but how does the calculator do it? If we had a power series representation for $\sin x$ and we knew it converged for the value of x we had in mind, we couldn't necessarily write down the limit but we could calculate partial sums to get an estimation as accurate as we like.

Questions: What functions can be represented by power series, and on what sorts of interval or subsets of \mathbb{R} ? If a function could be represented by a power series, how would we calculate the coefficients in this series?

We are not going to give a full answer to these questions, but a partial one involving *Maclaurin* or *Taylor* series.

Suppose that $f(x)$ is an infinitely differentiable function (this means that all the derivatives of f are themselves differentiable), and suppose that f is represented by the power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

We can work out appropriate values for the coefficients c_n as follows.

- Put $x = 0$. Then $f(0) = c_0 + \sum_{n=1}^{\infty} c_n (0)^n \implies f(0) = c_0$.
The constant term in the power series is the value of f at 0.
- To calculate c_1 , look at the value of the first derivative of f at 0, and differentiate the power series term by term. We expect

$$f'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

Then we should have

$$f'(0) = c_1 + 2c_2 \times 0 + 3c_3 \times 0 + \dots = c_1.$$

Thus $c_1 = f'(0)$.

- For c_2 , look at the second derivative of f . We expect

$$f''(x) = 2(1)c_2 + 3(2)c_3x + 4(3)c_4x^2 + 5(4)c_5x^3 + \dots = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Putting $x = 0$ gives $f''(0) = 2(1)c_2$ or $c_2 = \frac{f''(0)}{2(1)}$.

- For c_3 , look at the third derivative $f^{(3)}(x)$. We have

$$f^{(3)}(x) = 3(2)(1)c_3 + 4(3)(2)c_4x + 5(4)(3)c_5x^2 + \dots = \sum_{n=3}^{\infty} n(n-1)(n-2)c_n x^{n-3}.$$

Setting $x = 0$ gives $f^{(3)}(0) = 3(2)(1)c_3$ or $c_3 = \frac{f^{(3)}(0)}{3(2)(1)}$

Continuing this process, we obtain the following general formula for c_n :

$$c_n = \frac{1}{n!} f^{(n)}(0).$$

Definition 3.4.4. For a positive integer n , the number n factorial, denoted $n!$, is defined by

$$n! = n \times (n-1) \times (n-2) \times \dots \times 3 \times 2 \times 1.$$

The number $0!$ (zero factorial) is defined to be 1.

Example 3.4.5 (Power series representation of e^x).

The coefficient of x^n in the Maclaurin series expansion of e^x is

$$c_n = \frac{1}{n!} \frac{d^n}{dx^n} (e^x)|_{x=0} = \frac{1}{n!} e^0 = \frac{1}{n!}.$$

Thus the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Note that if we differentiate this series term by term we get exactly the same series back, which is what we would expect for a power series that represents e^x , since $\frac{d}{dx}(e^x) = e^x$.

Theorem 3.4.6. For every real number x , the above power series converges to e^x . The interval of convergence of this power series is all of \mathbb{R} and the radius of convergence is infinite.

Example 3.4.7 (Power series representation of $\sin x$).

Write $f(x) = \sin x$, and write $\sum_{n=0}^{\infty} c_n x^n$ for the Maclaurin series of $\sin x$. Then

- $f(0) = \sin 0 = 0 \implies c_0 = 0$
- $f'(0) = \cos 0 = 1 \implies c_1 = 1$
- $f''(0) = -\sin 0 = 0 \implies c_2 = \frac{0}{2!} = 0$
- $f^{(3)}(0) = -\cos 0 = -1 \implies c_3 = \frac{-1}{3!} = -\frac{1}{6}$
- $f^{(4)}(0) = \sin 0 = 0 \implies c_4 = \frac{0}{4!} = 0$

This pattern continues: if k is even then $f^{(k)}(0) = \pm \sin 0 = 0$, so $c_k = 0$.

If k is odd and $k \equiv 1 \pmod{4}$ then $f^{(k)}(0) = \cos 0 = 1$ and $c_k = \frac{1}{k!}$.

If k is odd and $k \equiv 3 \pmod{4}$ then $f^{(k)}(0) = -\cos 0 = -1$ and $c_k = -\frac{1}{k!}$.

Thus the Maclaurin series for $\sin x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

Note that this series only involves odd powers of x - this is not surprising because \sin is an *odd function*; it satisfies $\sin(-x) = -\sin x$.

Theorem 3.4.8. For every real number x , the above series converges to $\sin x$.

Thus computing partial sums of this series gives us an effective way of approximating $\sin x$ for any real number x .

Exercise 3.4.9. Show that the Maclaurin series for $\cos x$ is given by

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

(Note that this coincides with what you would get by differentiating term-by-term the series for $\sin x$, which is not surprising since $\frac{d}{dx}(\sin x) = \cos x$.)

LEARNING OUTCOMES FOR SECTION 3.4

After studying this section you should be able to

- State the meaning of the term *power series*,
- Explain the concept of the *radius of convergence* of a power series,
- Calculate the coefficients in (an initial segment of) the Maclaurin series representation of a given function.

3.5 Exam advice and sample question for Chapter 3

Here is a “sample Question 3”. For this question you will need to be able to use the concept of convergence and demonstrate a clear understanding of what it means. You will need to be able to state and apply the main theoretical elements of Chapter 3, such as the Monotone Convergence Theorem and the statement that every convergent sequence is bounded.

3. (a) Give an example of
- a convergent sequence of real numbers;
 - a sequence of real numbers that is bounded and divergent;
 - a sequence of real numbers that is strictly monotonically increasing;
 - a sequence of real numbers that is convergent and is not monotonic.
- (b) A sequence (a_n) of real numbers is defined by

$$a_0 = 4, a_n = \sqrt{a_{n-1}^2 - 2a_{n-1} + 4} \text{ for } n \geq 1.$$

- Write down the first four terms of the sequence.
 - Show that the sequence is bounded below by 2.
 - Show that the sequence is monotonically decreasing.
 - State why it can be deduced that the sequence is convergent, and determine its limit.
- (c) Find the first four terms in the Maclaurin series of $\frac{1}{1-x}$.

Sample Solution:

- (a)
- (a_n) defined by $a_n = \frac{1}{n}$ for $n \geq 1$.
 - (a_n) defined by $a_n = (-1)^n$, for $n \geq 1$.
 - (a_n) defined by $a_n = n$, $n \geq 1$.
 - (a_n) defined by $a_n = (-1)^n \frac{1}{n}$, for $n \geq 1$.
- (b)
- $a_0 = 4, a_1 = \sqrt{12}, a_2 = \sqrt{16 - 2\sqrt{12}}, a_3 = \sqrt{16 - 2\sqrt{12} - 2\sqrt{16 - 2\sqrt{12} + 4}}$
 - Certainly $a_0 > 2$. Suppose that $a_k > 2$ for some k . Then $a_k^2 - 2a_k > 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} > \sqrt{4} \implies a_{k+1} > 2.$$

- iii We know that $a_k > 2$ for $k \in \mathbb{N}$. Then $4 - 2a_k < 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} < \sqrt{a_k^2} \implies a_{k+1} < a_k.$$

- iv Since the sequence (a_n) is bounded below and monotonically decreasing, it is convergent by the Monotone Convergence Theorem. Its limit L must satisfy

$$L = \sqrt{L^2 - 2L + 4} \implies L^2 = L^2 - 2L + 4 \implies 2L = 4 \implies L = 2.$$

- (c) Write $f(x) = \frac{1}{1-x}$. Then

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ \frac{1}{2}f''(0) &= \frac{1}{2}(2) = 1 \\ \frac{1}{3}f^{(3)}(0) &= \frac{1}{6}(6) = 1 \end{aligned}$$

First four terms of Maclaurin series: $1, x, x^2, x^3$.