3.5 Exam advice and sample question for Chapter 3

Here is a "sample Question 3". For this question you will need to be able to use the concept of convergence and demonstrate a clear understanding of what it means. You will need to be able to state and apply the main theoretical elements of Chapter 3, such as the Monotone Convergence Theorem and the statement that every convergent sequence is bounded.

3. (a) Give an example of
   i. a convergent sequence of real numbers;
   ii. a sequence of real numbers that is bounded and divergent;
   iii. a sequence of real numbers that is strictly monotonically increasing;
   iv. a sequence of real numbers that is convergent and is not monotonic.

   (b) A sequence (a_n) of real numbers is defined by

$$a_0 = 4,\ a_n = \sqrt{a_{n-1}^2 - 2a_{n-1} + 4} \text{ for } n \geqslant 1.$$

   i. Write down the first four terms of the sequence.
   ii. Show that the sequence is bounded below by 2.
   iii. Show that the sequence is montonically decreasing.
   iv. State why it can be deduced that the sequence is convergent, and determine its limit.

   (c) Find the first four terms in the Maclaurin series of $\dfrac{1}{1-x}$.

Sample Solution:

(a) i (a_n) defined by $a_n = \frac{1}{n}$ for $n \geqslant 1$.
ii (a_n) defined by $a_n = (-1)^n$, for $n \geqslant 1$.
iii (a_n) defined by $a_n = n,\ n \geqslant 1$.
iv (an) defined by $a_n = (-1)^n \frac{1}{n}$, for $n \geqslant 1$.

(b) i $a_0 = 4,\ a_1 = \sqrt{12},\ a_2 = \sqrt{16 - 2\sqrt{12}},\ a_3 = \sqrt{16 - 2\sqrt{12} - 2\sqrt{16 - 2\sqrt{12}} + 4}$
ii Certainly $a_0 > 2$. Suppose that $a_k > 2$ for some k. Then $a_k^2 - 2a_k > 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} > \sqrt{4} \Longrightarrow a_{k+1} > 2.$$

iii We know that $a_k > 2$ for $k \in \mathbb{N}$. Then $4 - 2a_k < 0$ and

$$a_{k+1} = \sqrt{a_k^2 - 2a_k + 4} < \sqrt{a_k^2} \Longrightarrow a_{k+1} < a_k.$$

iv Since the sequence (a_n) is bounded below and monotonically decreasing, it is convergent by the Monotone Convergence Theorem. Its limit L must satisfy

$$L = \sqrt{L^2 - 2L + 4} \Longrightarrow L^2 = L^2 - 2L + 4 \Longrightarrow 2L = 4 \Longrightarrow L = 2.$$

(c) Write $f(x) = \dfrac{1}{1-x}$. Then

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 1 \\ \frac{1}{2}f''(0) &= \frac{1}{2}(2) = 1 \\ \frac{1}{3}f^{(3)}(0) &= \frac{1}{6}(6) = 1 \end{aligned}$$

First four terms of Maclaurin series: $1,\ x,\ x^2, x^3$.