Section 1.4 Techniques of Integration

To calculate

$$\int_a^b f(x)dx$$

1. Find a function F for which $F'(x) = f(x)$, i.e. find a function F whose derivative is f.
2. Evaluate F at the limits of integration a and b; i.e. calcuate $F(a)$ and $F(b)$. This means replacing x separately with a and b in the formula that defines $F(x)$.
3. Calculate the number $F(b) - F(a)$. This is the definite integral $\displaystyle\int_a^b f(x)\,dx$.

Of the three steps above, the first one is the hard one.

Notation

Recall the following notation : if F is a function that satisfies $F'(x) = f(x)$, then

$$F(x)\Big|_a^b \text{ or } F(x)\Big|_{x=a}^{x=b} \text{ means } F(b) - F(a).$$

Definition 14

Let f be a function. Another function F is called an antiderivative of f if the derivative of F is f, i.e. if $F'(x) = f(x)$, for all (relevant) values of the variable x.

So for example x^2 is an antiderivative of $2x$. Note that $x^2 + 1$, $x^2 + 5$ and $x^2 - 20e$ are also antiderivatives of $2x$. So we talk about an antiderviative of a function or expression rather that the antiderivative.

Definition 15

Let f be a function. The indefinite integral of f, written

$$\int f(x)\,dx$$

is the "general antiderivative" of f. If $F(x)$ is a particular antiderivative of f, then we would write

$$\int f(x)\,dx = F(x) + C,$$

to indicate that the different antiderivatives of f look like $F(x) + C$, where C may be any constant. (In this context C is often referred to as a constant of integration).

Example 16

Determine $\int \cos 2x\, dx$.

Solution: The question is: what do we need to differentiate to get $\cos 2x$?
Well, what do we need to differentiate to get something involving cos?
The derivative of $\sin x$ is $\cos x$. A reasonable guess would say that the derivative of $\sin 2x$ might be "something like" $\cos 2x$.
By the chain rule, the derivative of $\sin 2x$ is in fact $2\cos 2x$.
So $\sin 2x$ is pretty close but it gives us twice what we want - we should compensate for this by taking $\frac{1}{2}\sin 2x$; its derivative is

$$\frac{1}{2}(2\cos 2x) = \cos 2x.$$

Conclusion: $\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$

Powers of x

Example 17

Determine $\int x^n \, dx$

Important Note: We know that in order to calculate the derivative of an expression like x^n, we reduce the index by 1 to $n-1$, and we multiply by the constant n. So

$$\frac{d}{dx}x^n = nx^{n-1}$$

in general. To find an antiderivative of x^n we have to reverse this process. This means that the index increases by 1 to $n+1$ and we multiply by the constant $\frac{1}{n+1}$. So

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.$$

This makes sense as long as the number n is not equal to -1 (in which case the fraction $\frac{1}{n+1}$ wouldn't be defined).

The Integral of $\frac{1}{x}$

Suppose that $x > 0$ and $y = \ln x$. Recall this means (by definition) that $e^y = x$. Differentiating both sides of this equation (with respect to x) gives

$$e^y \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

Thus the derivative of $\ln x$ is $\frac{1}{x}$, and

$$\int \frac{1}{x}\, dx = \ln x + C, \text{ for } x > 0.$$

If $x < 0$, then

$$\int \frac{1}{x}\, dx = \ln |x| + C.$$

This latter formula applies for all $x \neq 0$.

A definite integral

Example 18

Determine $\int_0^{\pi} \sin x + \cos x \, dx$.

Solution: We need to write down *any* antiderivative of $\sin x + \cos x$ and evaluate it at the limits of integration :

$$\begin{aligned} \int_0^{\pi} \sin x + \cos x \, dx &= -\cos x + \sin x\Big|_0^{\pi} \\ &= (-\cos \pi + \sin \pi) - (-\cos 0 + \sin 0) \\ &= -(-1) + 0 - (-1 + 0) = 2. \end{aligned}$$

Note: To determine $\cos \pi$, start at the point $(1, 0)$ and travel counter–clockwise around the unit circle through an angle of π radians (180 degrees), arriving at the point $(-1, 0)$. The x-coordinate of the point you are at now is $\cos \pi$, and the y-coordinate is $\sin \pi$.