1.5 Consistent and Inconsistent Systems

Example 1.5.1 *Consider the following system :*

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & 5z & = & 4 \\ x & + & y & - & 2z & = & 1 \\ 5x & + & 3y & - & 8z & = & 6 \end{array}$$

To find solutions, obtain a row-echelon form from the augmented matrix :

$$\begin{pmatrix} 3 & 2 & -5 & 4 \\ 1 & 1 & -2 & 1 \\ 5 & 3 & -8 & 6 \end{pmatrix} \quad \begin{array}{c} R1 \leftrightarrow R2 \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & -5 & 4 \\ 5 & 3 & -8 & 6 \end{pmatrix}$$

$$\begin{array}{c} R2 \to R2 - 3R1 \\ \longrightarrow \\ R3 \to R3 - 5R1 \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \quad \begin{array}{c} R2 \times (-1) \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & 2 & 1 \end{pmatrix}$$

$$\begin{array}{c} R3 \to R3 + 2R2 \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \begin{array}{c} R3 \times (-1) \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Row-Echelon Form)

The system of equations corresponding to this REF has as its third equation

$$0x + 0y + 0z = 1 \quad \text{i.e.} \quad 0 = 1$$

This equation clearly has no solutions - no assignment of numerical values to x, y and z will make the value of the expression $0x + 0y + 0z$ equal to anything but zero. Hence the system has no solutions.

Definition 1.5.2 *A system of linear equations is called* inconsistent *if it has no solutions. A system which has a solution is called* consistent.

If a system is inconsistent, a REF obtained from its augmented matrix will include a row of the form 0 0 0 ... 0 1, i.e. will have a leading 1 in its rightmost column. Such a row corresponds to an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = 1$, which certainly has no solution.

Example 1.5.3 *(MA203 Summer 2005, Q1)*

(a) Find the unique value of t for which the following system has a solution.

$$\begin{array}{rcrcrcrcr} -x_1 & & & + & x_3 & - & x_4 & = & 3 \\ 2x_1 & + & 2x_2 & - & x_3 & - & 7x_4 & = & 1 \\ 4x_1 & - & x_2 & - & 9x_3 & - & 5x_4 & = & t \\ 3x_1 & - & x_2 & - & 8x_3 & - & 6x_4 & = & 1 \end{array}$$

Solution: First write down the augmented matrix and begin Gauss-Jordan elimination.

$$\begin{pmatrix} -1 & 0 & 1 & -1 & 3 \\ 2 & 2 & -1 & -7 & 1 \\ 4 & -1 & -9 & -5 & t \\ 3 & -1 & -8 & -6 & 1 \end{pmatrix} \quad \begin{array}{c} R1 \times (-1) \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 2 & 2 & -1 & -7 & 1 \\ 4 & -1 & -9 & -5 & t \\ 3 & -1 & -8 & -6 & 1 \end{pmatrix}$$

$$\begin{array}{c} R2 \to R2 - 2R1 \\ R3 \to R3 - 4R1 \\ \longrightarrow \\ R4 \to R4 - 3R1 \end{array} \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & -1 & -5 & -9 & t+12 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix} \quad \begin{array}{c} R3 \to R3 - R4 \\ \longrightarrow \end{array} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & 0 & 0 & 0 & t+2 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix}$$

From the third row of this matrix we can see that the system can be consistent only if $t+2=0$. i.e. only if $t=-2$.

(b) Find the general solution of this system for this value of t.

Solution: Set $t=-2$ and continue with the Gaussian elimination. We omit the third row, which consists fully of zeroes and carries no information.

$$\begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 2 & 1 & -9 & 7 \\ 0 & -1 & -5 & -9 & 10 \end{pmatrix} \quad \overset{R4\times(-1)}{\underset{\mathbb{R}3 \leftrightarrow R4}{\longrightarrow}} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 2 & 1 & -9 & 7 \end{pmatrix}$$

$$\overset{R3\to R3-2R2}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 0 & -9 & -27 & 27 \end{pmatrix} \quad \overset{R3\times(-\frac{1}{9})}{\longrightarrow} \quad \begin{pmatrix} 1 & 0 & -1 & 1 & -3 \\ 0 & 1 & 5 & 9 & -10 \\ 0 & 0 & 1 & 3 & -3 \end{pmatrix}$$

$$\overset{R1\to R1+R3}{\underset{R2\to R2+5R3}{\longrightarrow}} \quad \begin{pmatrix} 1 & 0 & 0 & 4 & -6 \\ 0 & 1 & 0 & -6 & 5 \\ 0 & 0 & 1 & 3 & -3 \end{pmatrix}$$

Having reached a reduced row-echelon form, we can see that the variables $x_1,\ x_2$ and x_3 are leading variables, and the variable x_4 is free. We have from the RREF

$$x_1 = -6 - 4x_4,\ x_2 = 5 + 6x_4,\ x_3 = -3 - 3x_4.$$

If we assign the parameter name s to the value of the free variable x_4 in a solution of the system, we can write the general solution as

$$(x_1, x_2, x_3, x_4) = (-6 - 4s, 5 + 6s, -3 - 3s, s),\ s \in \mathbb{R}.$$

Summary of Possible Outcomes when Solving a System of Linear Equations:

1. The system may be inconsistent. This happens if a REF obtained from the augmented matrix has a leading 1 in its rightmost column.

2. The system may be consistent. In this case one of the following occurs :

   (a) There may be a unique solution. This will happen if all variables are leading variables, i.e. every column except the rightmost one in a REF obtained from the augmented matrix has a leading 1. In the case the *reduced* row-echelon form obtained from the augmented matrix will have the following form :

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with possibly some additional rows full of zeroes at the bottom. The unique solution can be read from the right-hand column.

Note: If a system of equations has a unique solution, the number of equations must be at least equal to the number of variables (since the augmented matrix must have enough rows to accommodate a leading 1 for every variable).

(b) There may be infinitely many solutions. This happens if the system is consistent but at least one of the variables is free. In this case the rank of the augmented matrix will be less than the number of variables in the system.